CS255: Winter 2011

PRPs and PRFs

1. Abstract ciphers: PRPs and PRFs,
2. Security models for encryption,
3. Analysis of CBC and counter mode

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PRPs and PRFs

- Pseudo Random Function (**PRF**) defined over (K,X,Y):

  $$F:\ K \times X \rightarrow Y$$

  such that exists "efficient" algorithm to evaluate F(k,x)

- Pseudo Random Permutation (**PRP**) defined over (K,X):

  $$E:\ K \times X \rightarrow X$$

  such that:

  1. Exists "efficient" algorithm to evaluate E(k,x)
  2. The function E(k, ·) is one-to-one
  3. Exists "efficient" inversion algorithm D(k,x)

Running example

- Example PRPs: 3DES, AES, …

  AES: $K \times X \rightarrow X$ where $K = X = \{0,1\}^{128}$

  DES: $K \times X \rightarrow X$ where $X = \{0,1\}^{64}$, $K = \{0,1\}^{56}$

  3DES: $K \times X \rightarrow X$ where $X = \{0,1\}^{64}$, $K = \{0,1\}^{168}$

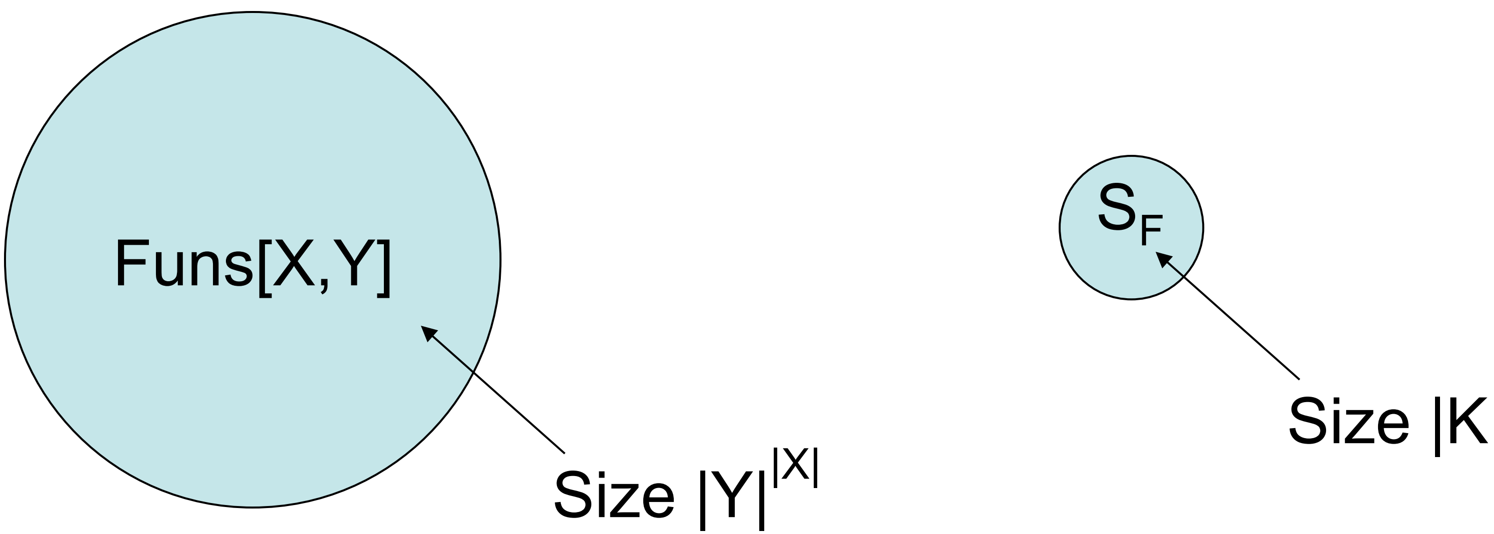
- Functionally, any PRP is also a PRF.
  – A PRP is a PRF where X=Y and is efficiently invertible.

Secure PRFs

- Let F: $K \times X \rightarrow Y$ be a PRF

  - Funs[X,Y]: the set of **<u>all</u>** functions from X to Y
  - S_F = { F(k,·) s.t. $k \in K$ } $\subseteq$ Funs[X,Y]

- <u>Intuition</u>: a PRF is **secure** if
  a random function in Funs[X,Y] is indistinguishable from
  a random function in S_F



Secure PRFs

- Let $F: K \times X \rightarrow Y$ be a PRF

  $\begin{cases} \text{Funs}[X,Y]: & \text{the set of } \textbf{all} \text{ functions from X to Y} \\ S_F = \{ F(k,\cdot) \ \text{ s.t. } \ k \in K \} & \subseteq \ \text{Funs}[X,Y] \end{cases}$

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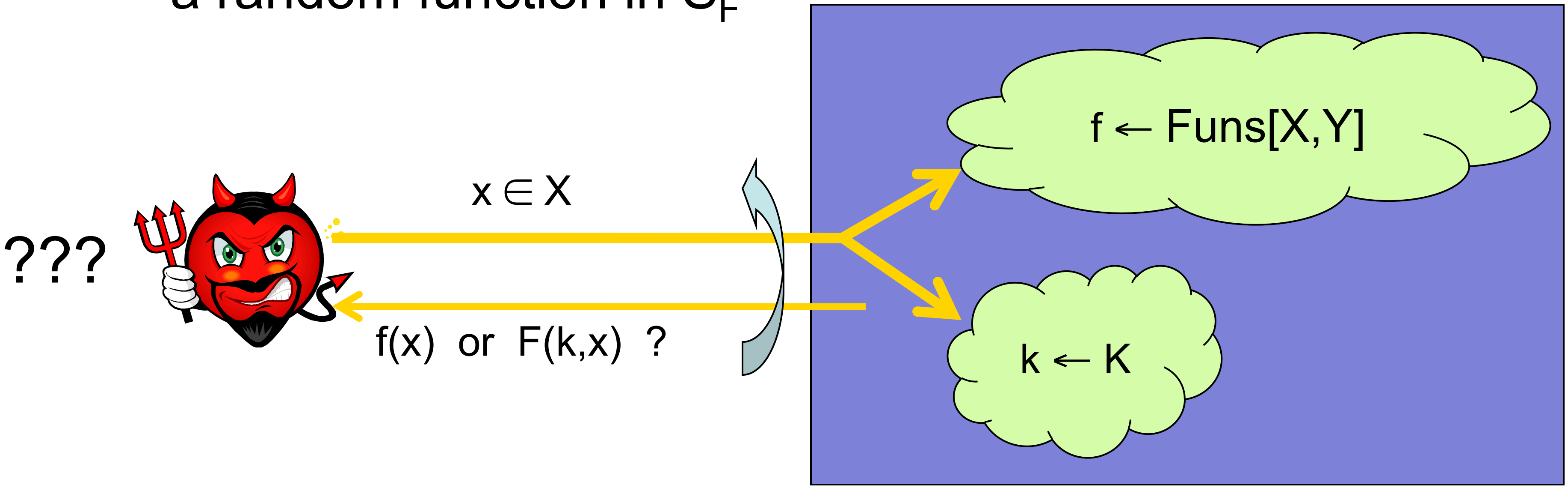
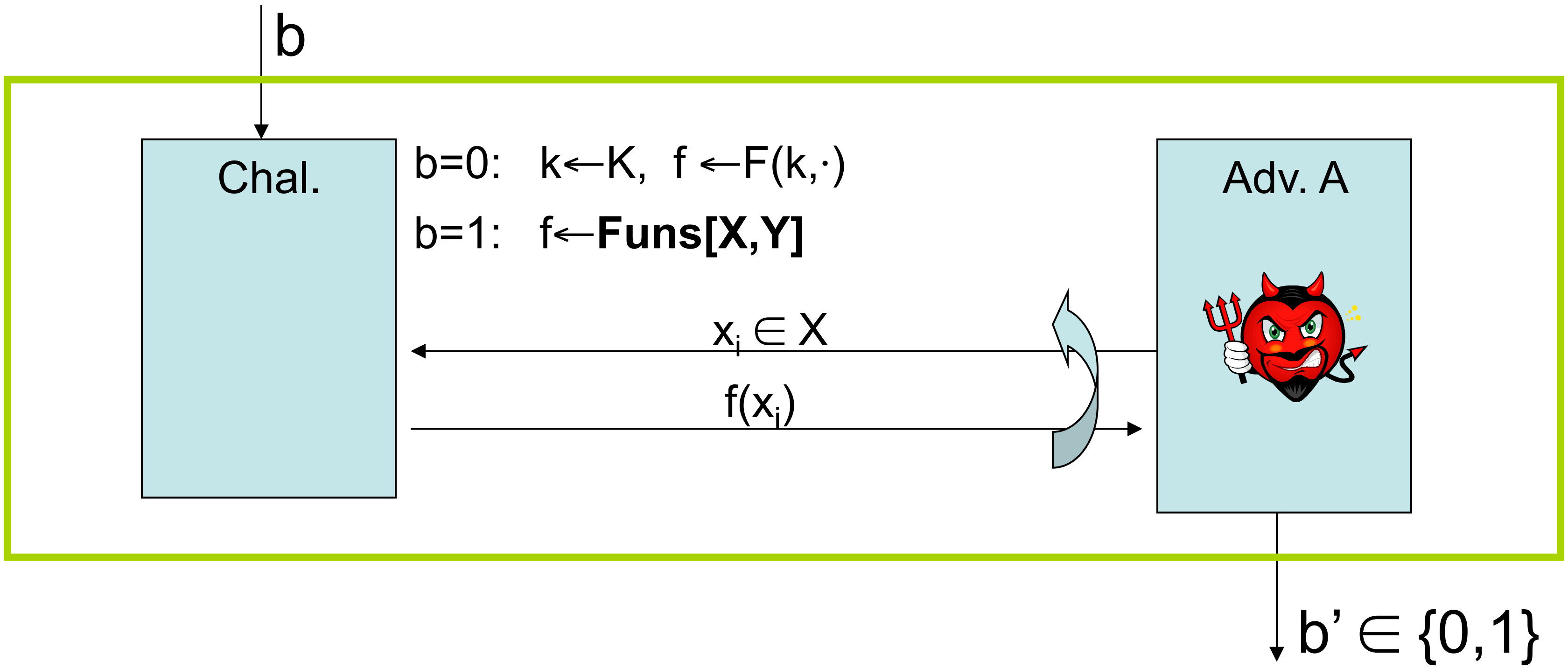


Secure PRF: defintion

- For b=0,1 define experiment EXP(b) as:



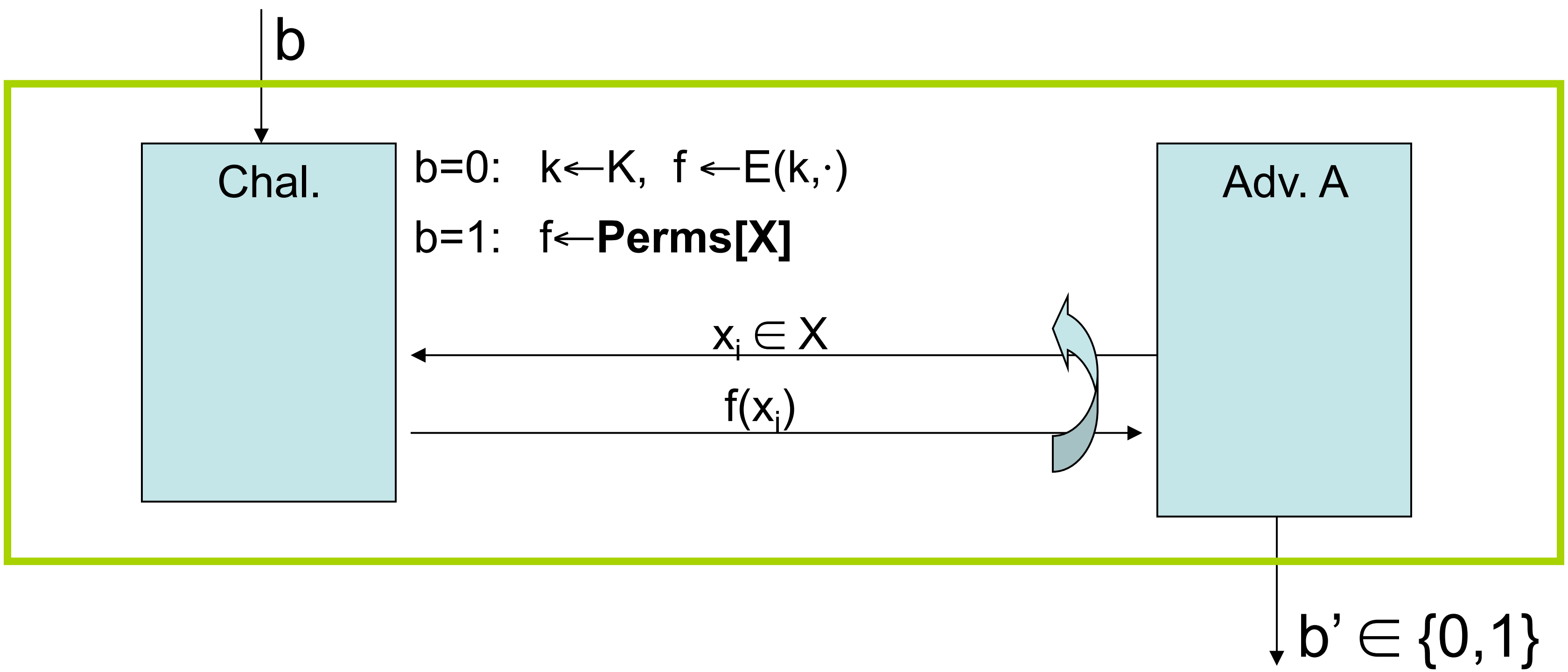
- Def: F is a secure PRF if for all "efficient" A:

$$\text{PRF Adv}[A,F] = \left| \Pr[\text{EXP}(0)=1] - \Pr[\text{EXP}(1)=1] \right|$$

is "negligible."

Secure PRP

- For b=0,1 define experiment EXP(b) as:



- Def: E is a secure PRP if for all "efficient" A:

$$\text{PRP Adv}[A,E] = \left| \Pr[\text{EXP}(0)=1] - \Pr[\text{EXP}(1)=1] \right|$$

is "negligible."

Example secure PRPs

- Example secure PRPs: 3DES, AES, …

  AES: $K \times X \rightarrow X$ where $K = X = \{0,1\}^{128}$

- AES PRP Assumption (example) :

  All 2^{80}–time algs A have PRP Adv[A, **AES**] $< 2^{-40}$

PRF Switching Lemma

- Any secure PRP is also a secure PRF.

- <u>Lemma</u>: Let E be a PRP over (K,X)

  Then for any q-query adversary A:

$$\left| \text{PRF Adv}[A,E] - \text{PRP Adv}[A,E] \right| < q^2 / 2|X|$$

⇒ Suppose |X| is large so that $q^2 / 2|X|$ is "negligible"

Then

PRP Adv[A,E] "negligible" ⇒ PRF Adv[A,E] "negligible"

Using PRPs and PRFs

- <u>Goal</u>: build "secure" encryption from a PRP.

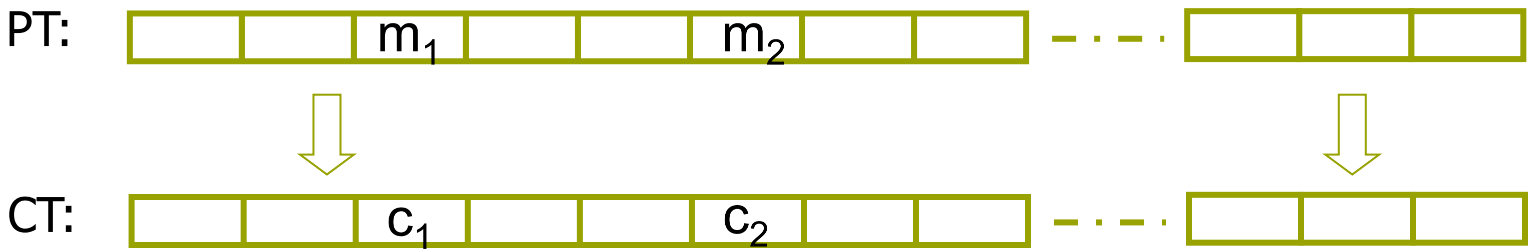
- Security is always defined using two parameters:

  1. What "**power**" does adversary have?
     examples:
     - Adv sees only one ciphertext (one-time key)
     - Adv sees many PT/CT pairs (many-time key, CPA)

  2. What "**goal**" is adversary trying to achieve?
     examples:
     - Fully decrypt a challenge ciphertext.
     - Learn info about PT from CT (semantic security)

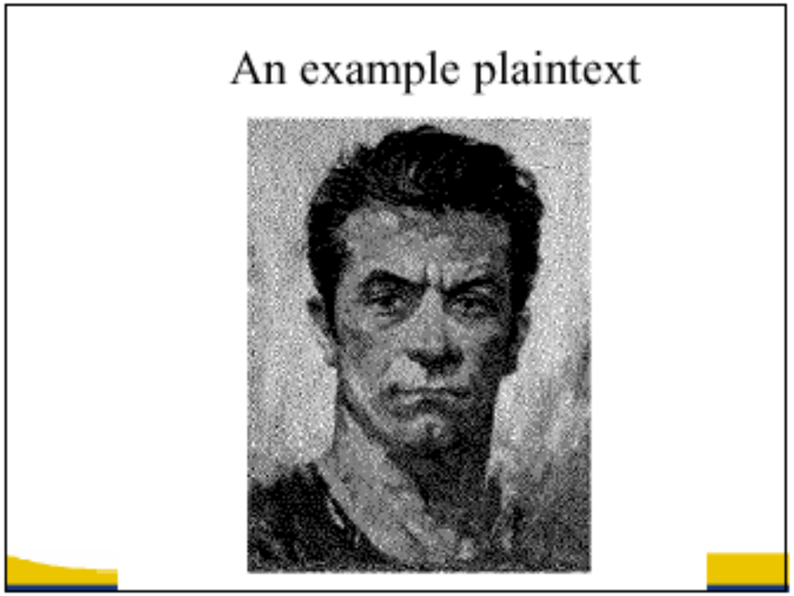
Incorrect use of a PRP

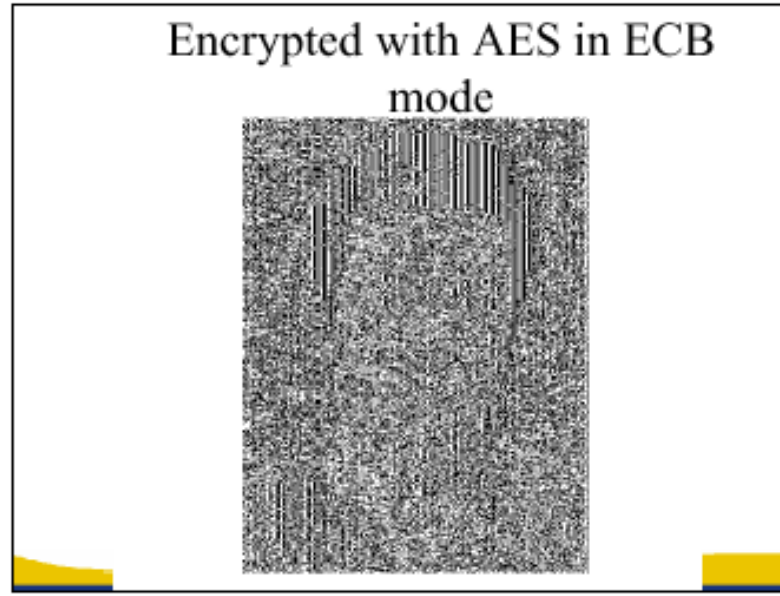
Electronic Code Book (ECB):



- Problem:
  - if $m_1=m_2$ then $c_1=c_2$

In pictures





(courtesy B. Preneel)

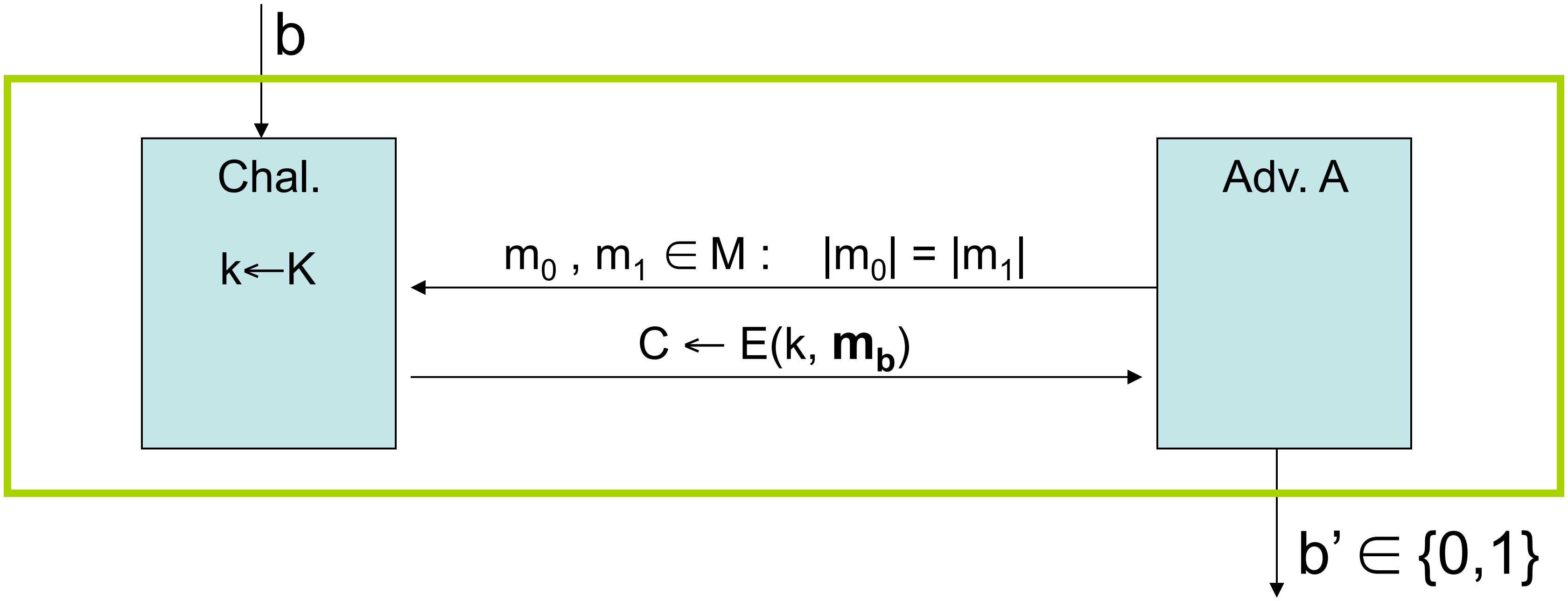
Modes of Operation for One-time Use Key

Example application:

Encrypted email. New key for every message.

Semantic Security for one-time key

- E = (E,D)   a cipher defined over (K,M,C)
- For   b=0,1   define EXP(b) as:



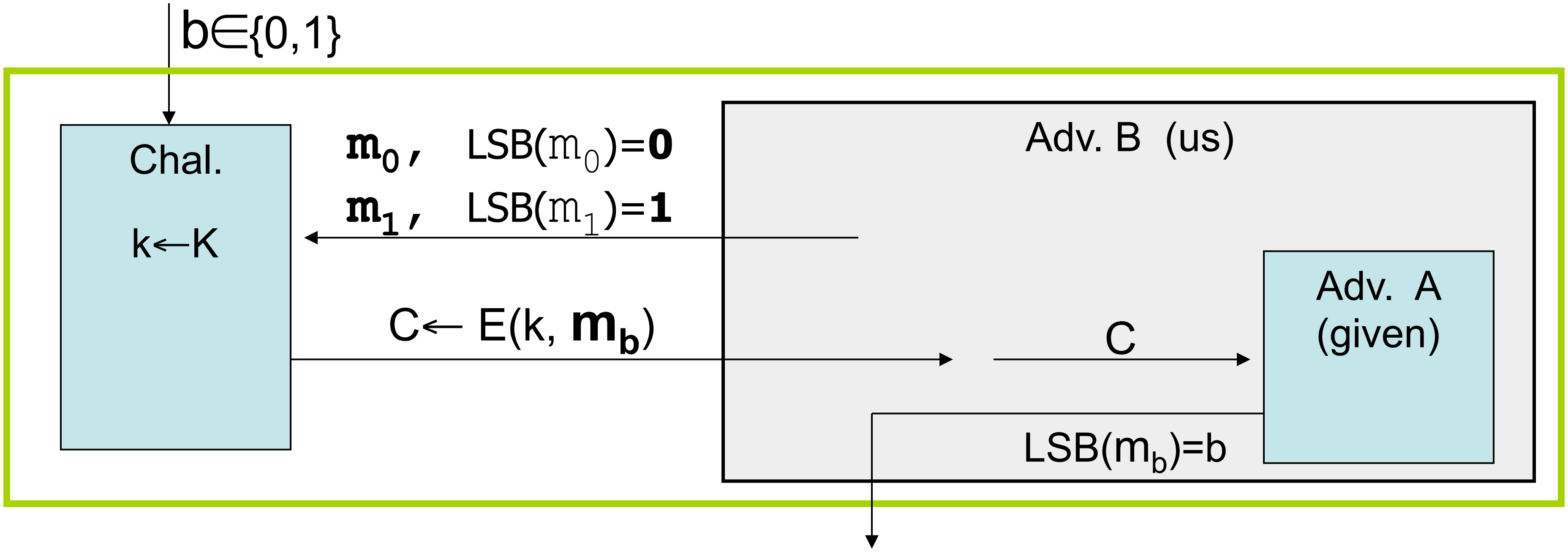
- Def: E is sem. sec. for one-time key if for all "efficient" A:

$$\text{SS Adv}[A,E] = \left| \Pr[\text{EXP}(0)=1] - \Pr[\text{EXP}(1)=1] \right|$$

is "negligible."

Semantic security (cont.)

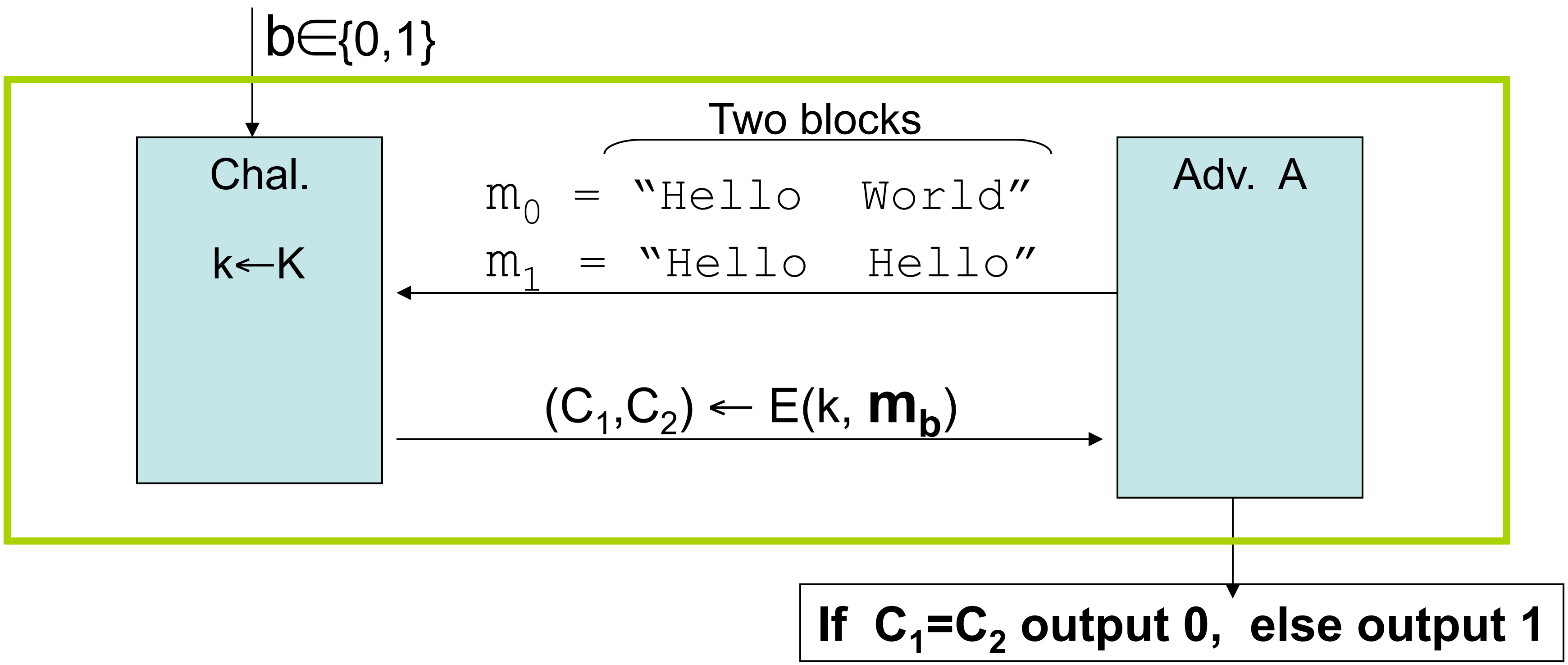
- Sem. Sec. ⇒ no "efficient" adversary learns info about PT from a **single** CT.
- Example: suppose efficient A can deduce LSB of PT from CT. Then E = (E,D) is not semantically secure.



- Then SS Adv[B, E] = 1 ⇒ E is not sem. sec.

Note: ECB is not Sem. Sec.

- Electronic Code Book (ECB):
  - Not semantically secure for messages that contain more than one block.



- Then SS Adv[A, ECB] = 1

Secure Constructions

- Examples of sem. sec. systems:
  1. SS Adv[A, OTP] = 0 for **all** A
  2. Deterministic counter mode from a PRF F :
     - E_{DETCTR} (k,m) =

     - Stream cipher built from PRF (e.g. AES, 3DES)

Det. counter-mode security

- Theorem: For any L>0.

  If F is a secure PRF over (K,X,X) then

  E_{DETCTR} is sem. sec. cipher over (K,X^L,X^L).

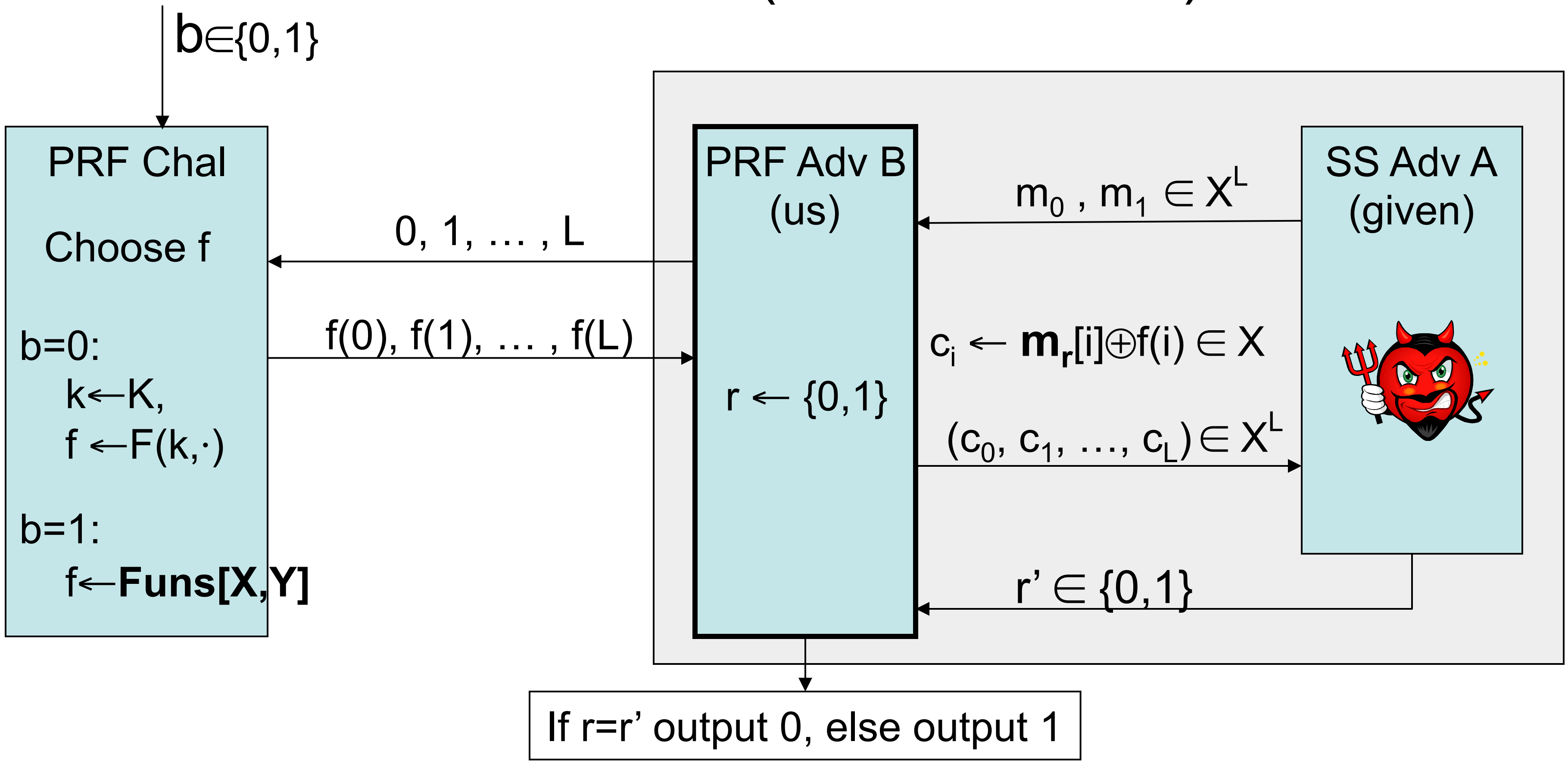
  In particular, for any adversary A attacking E_{DETCTR} there exists a PRF adversary B s.t.:

$$\text{SS Adv}[A, E_{DETCTR}] = 2\cdot\text{PRF Adv}[B, F]$$

PRF Adv[B, F] is negligible (since F is a secure PRF)

Hence, SS Adv[A, E_{DETCTR}] must be negligible.

Proof (as a reduction)



b=1: f←**Funs[X,X]** ⇒ Pr[EXP(1)=0] = Pr[r=r'] = ½

b=0: f←F(k,·) ⇒ Pr[EXP(0)=0] = ½ ± ½ ·SS Adv[A, E_{DETCTR}]

Hence, PRF Adv[F, B] = ½ ·SS Adv[A, DETCTR]

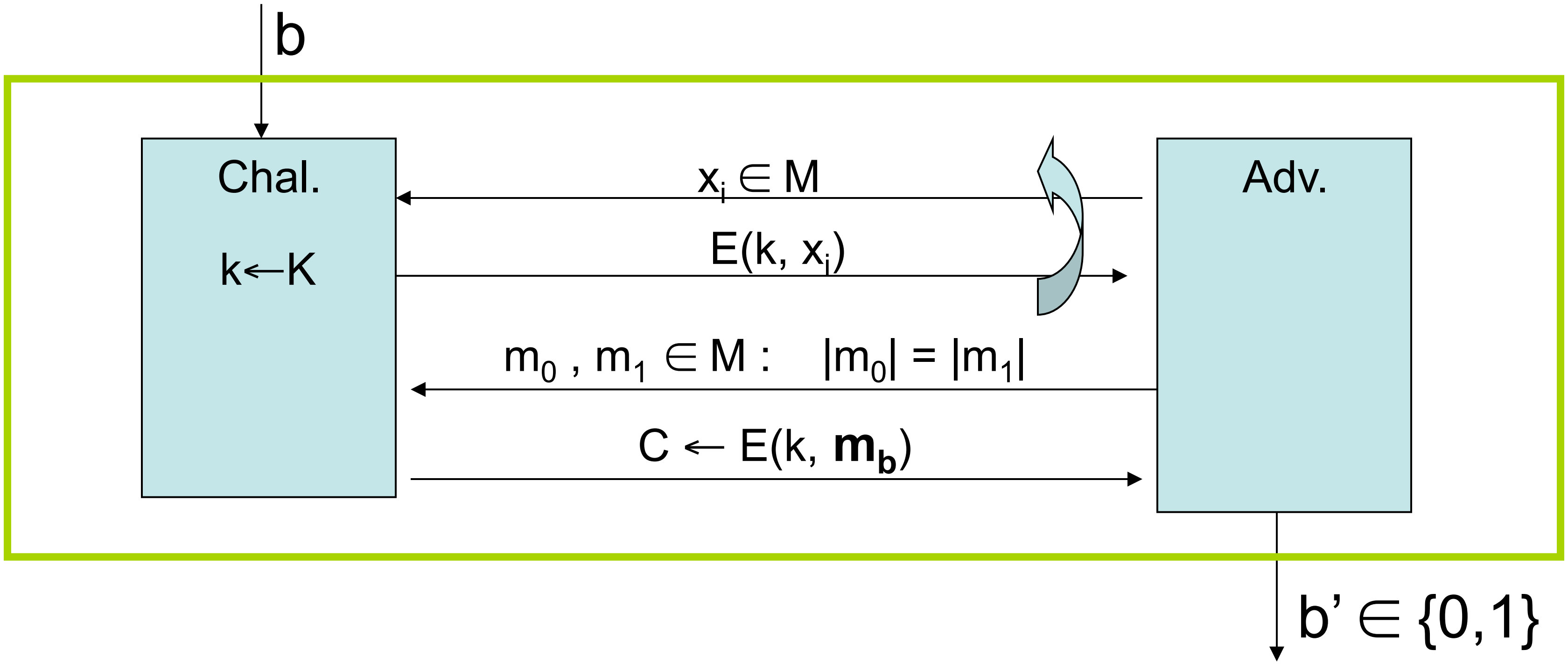
Modes of Operation for Many-time Key

Example applications:

1. File systems: Same AES key used to encrypt many files.
2. IPsec: Same AES key used to encrypt many packets.

Semantic Security for many-time key

- E = (E,D) a cipher defined over (K,M,C)
- For b=0,1 define EXP(b) as: (simplified CPA)



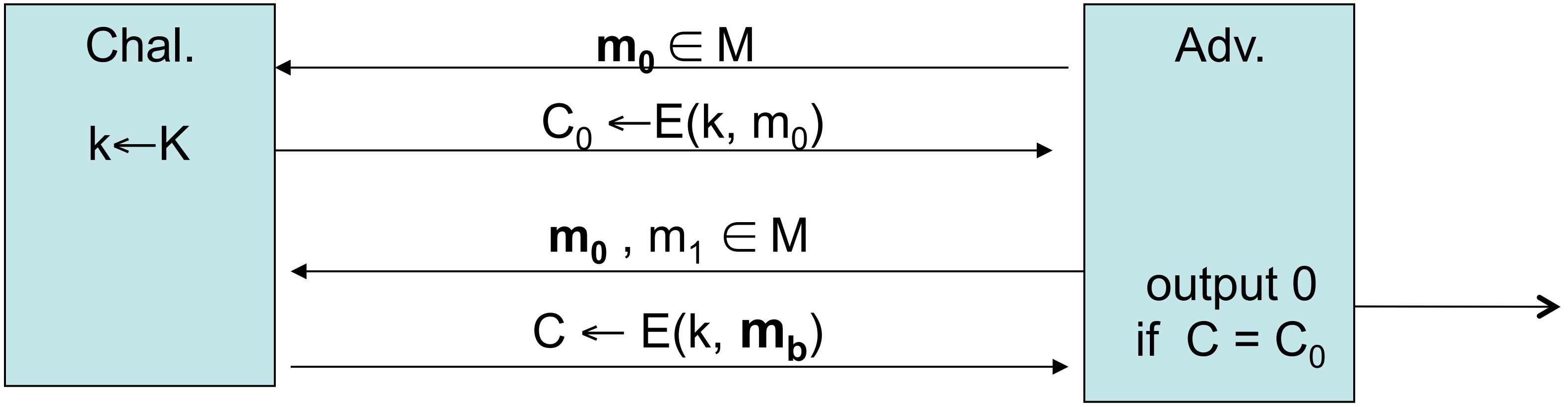
- Def: E is sem. sec. under CPA if for all “efficient” A:

$$SS^{CPA} Adv[A,E] = \left| Pr[EXP(0)=1] - Pr[EXP(1)=1] \right|$$

is “negligible.”

Security for many-time key

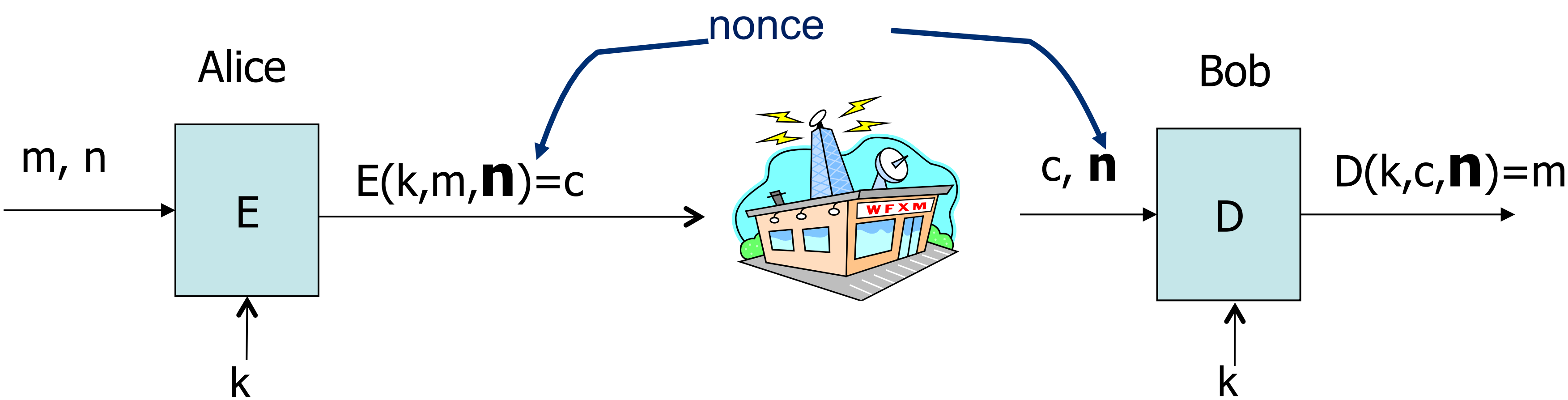
- <u>Fact:</u> stream ciphers are insecure under CPA.
  - More generally: if E(k,m) always produces same ciphertext, then cipher is insecure under CPA.



- If secret key is to be used multiple times ⇒

  given the same plaintext message twice,
  the encryption alg. must produce different outputs.

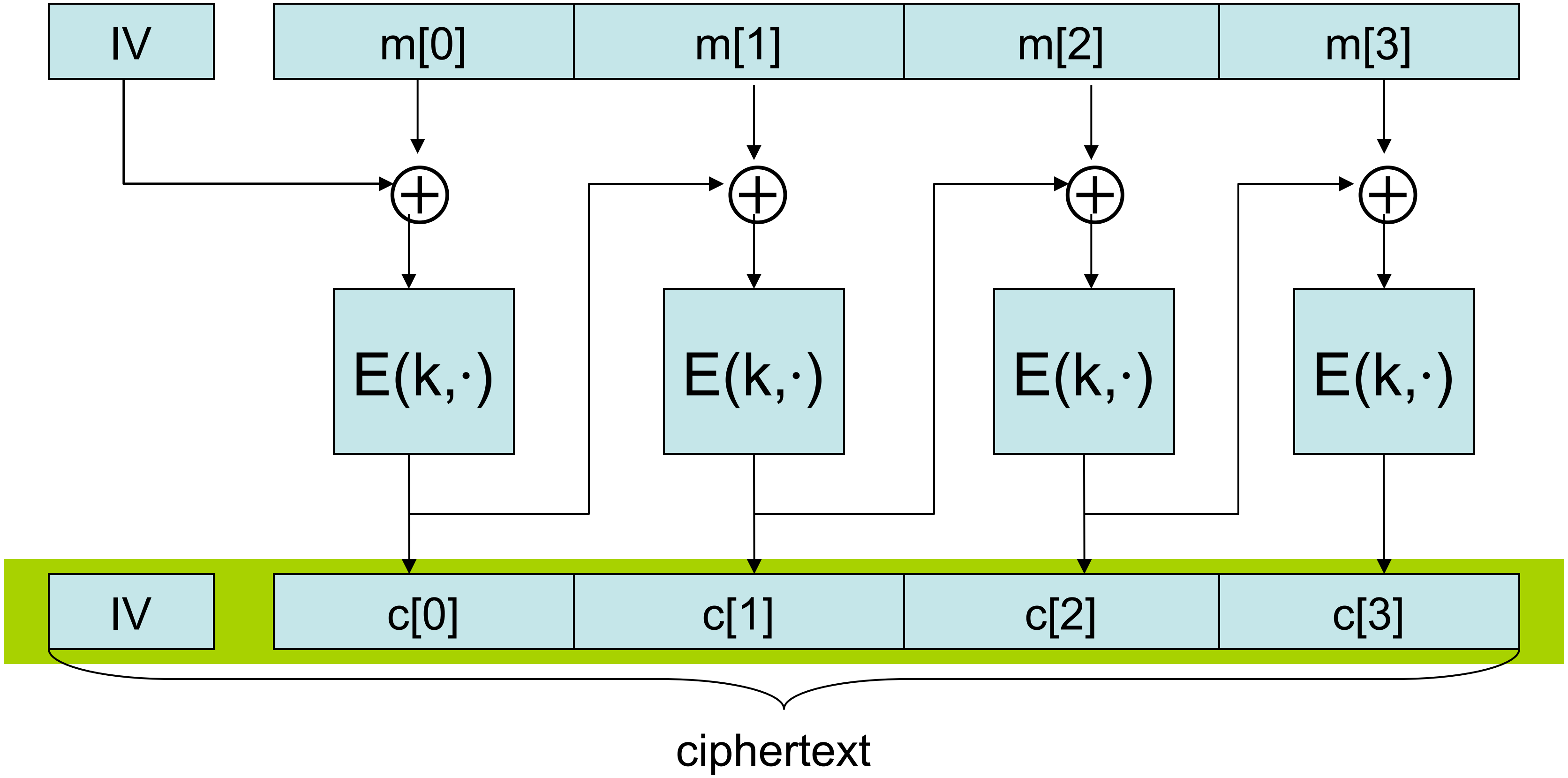
Nonce-based Encryption



- nonce n: a value that changes from msg to msg
  (k,n) pair <u>never</u> used more than once
- <u>method 1</u>: encryptor picks a random nonce, $n \leftarrow \mathcal{N}$
- <u>method 2</u>: nonce is a counter (e.g. packet counter)
  - used when encryptor keeps state from msg to msg
  - if decryptor has same state, need not send nonce with CT

Construction 1: CBC with random nonce

- Cipher block chaining with a <u>random</u> IV (IV = nonce)



note: CBC where attacker can predict the IV is not CPA-secure. HW.

CBC: CPA Analysis

- <u>CBC Theorem</u>: For any $L>0$,

  If E is a secure PRP over (K,X) then

  E_{CBC} is a sem. sec. under CPA over (K, X^L, X^{L+1}).

  In particular, for a q-query adversary A attacking E_{CBC} there exists a PRP adversary B s.t.:

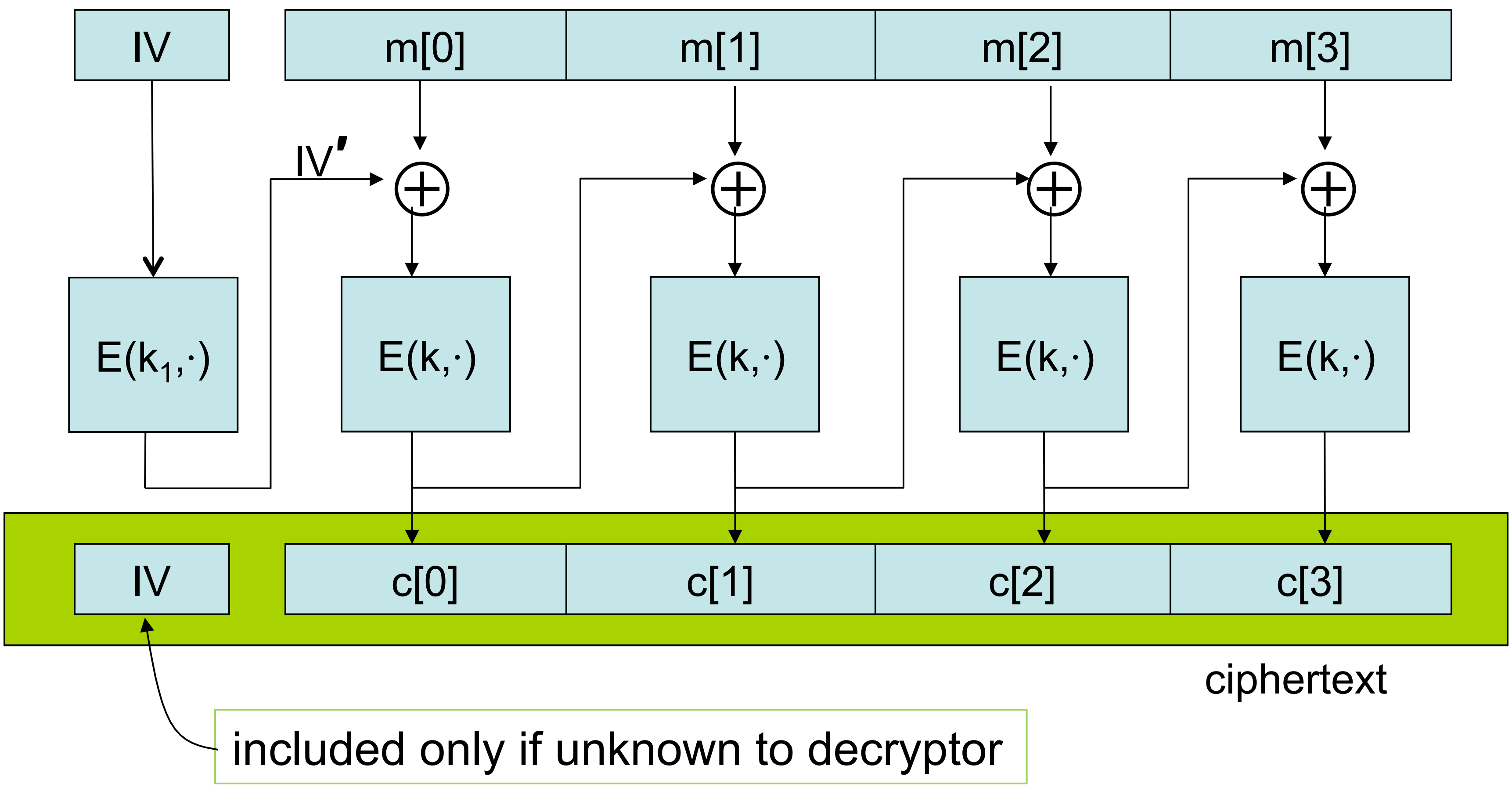
$$SS_{CPA}\ Adv[A, E_{CBC}] \le 2 \cdot PRP\ Adv[B, E] + 2\, q^2 L^2 / |X|$$

- Note: CBC is only secure as long as $q^2 L^2 \ll |X|$

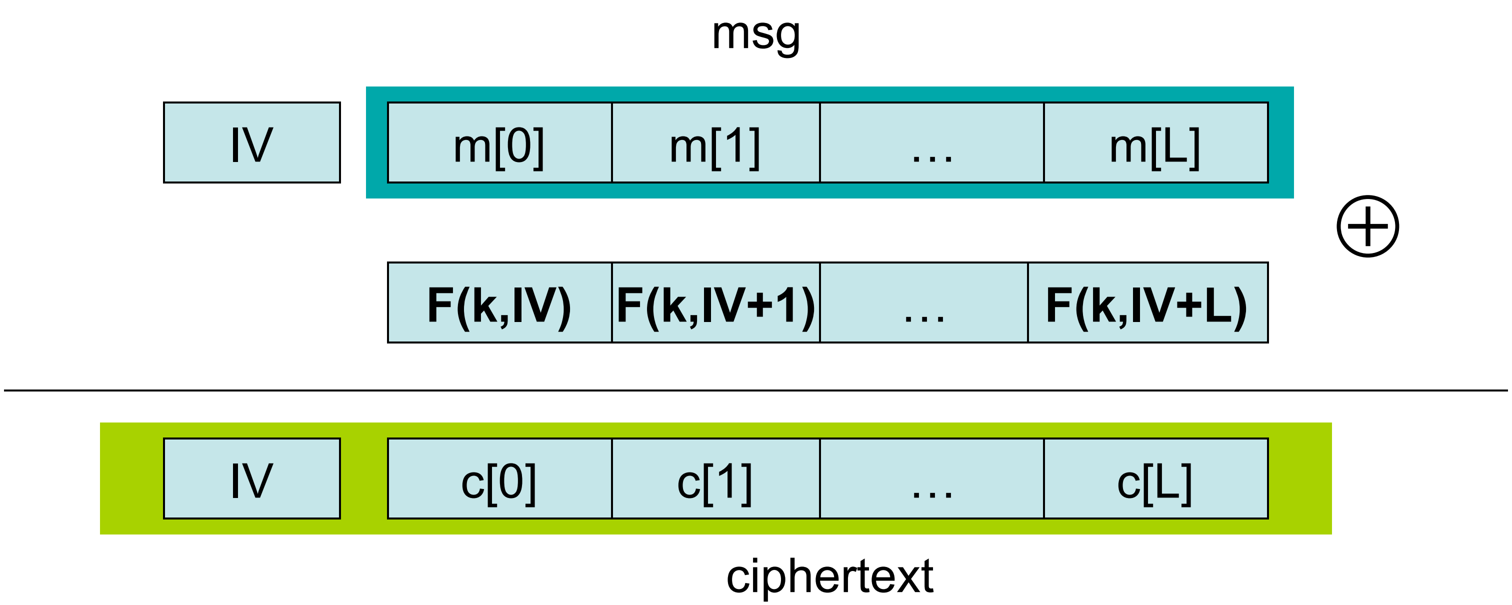
Construction 1': CBC with **unique** nonce

- Cipher block chaining with unique IV (IV = nonce)

unique IV means: (key,IV) pair is used for only one message



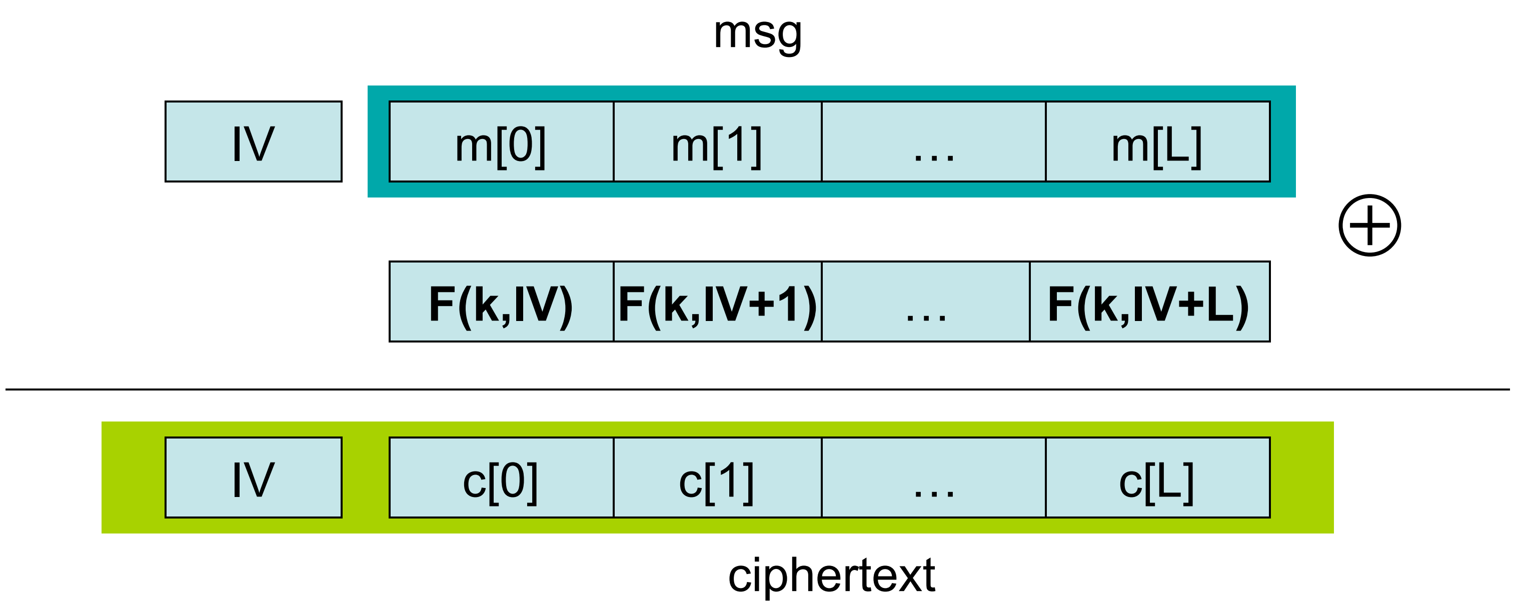
Construction 2: rand ctr-mode



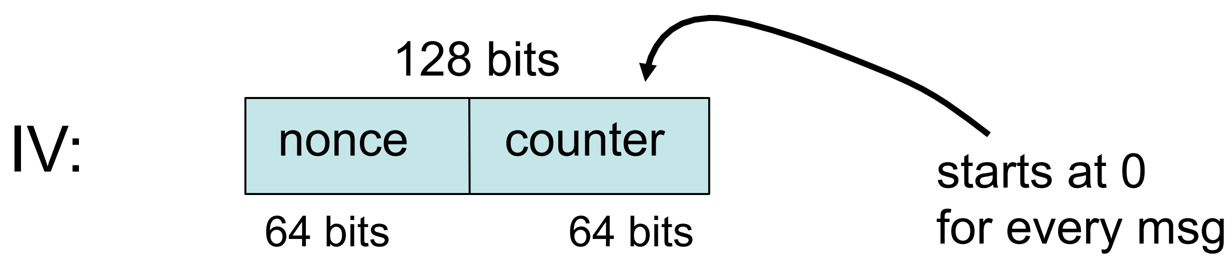
IV - chosen at random for every message

note: parallelizable (unlike CBC)

Construction 2': nonce ctr-mode



To ensure F(K,x) is never used more than once, choose IV as:



rand ctr-mode: CPA analysis

- Randomized counter mode: random IV.

- <u>Counter-mode Theorem</u>: For any L>0,

  If F is a secure PRF over (K,X,X) then

  E_{CTR} is a sem. sec. under CPA over (K,X^L,X^{L+1}).

  In particular, for a q-query adversary A attacking E_{CTR} there exists a PRF adversary B s.t.:

$$SS_{CPA}\ Adv[A, E_{CTR}] \le 2 \cdot PRF\ Adv[B, F] + 2\, q^2 L / |X|$$

- <u>Note</u>: ctr-mode only secure as long as $q^2L \ll |X|$

  Better then CBC !

Summary

- PRPs and PRFs: a useful abstraction of block ciphers.
- We examined two security notions:

  1. Semantic security against one-time CPA.
  2. Semantic security against many-time CPA.

  Note: neither mode ensures data integrity.
- Stated security results summarized in the following table:

Goal \ Power	**one-time key**	**Many-time key (CPA)**	**CPA and CT integrity**
Sem. Sec.	steam-ciphers det. ctr-mode	rand CBC rand ctr-mode	later