CS255: Winter 2023

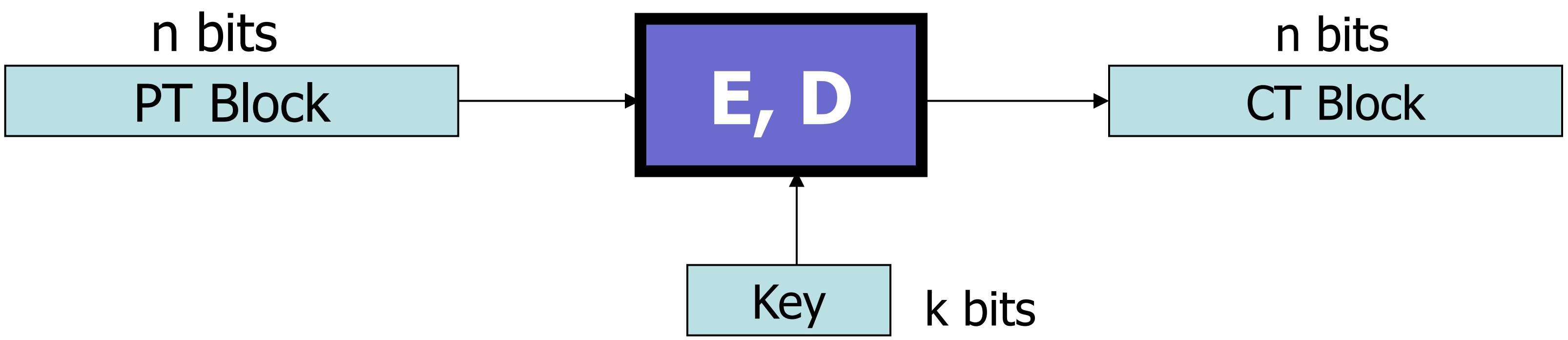
PRPs and PRFs

Final exam: March 22, 3 hours in a 24 hour window
(or 3:30pm in Gates B01)

Dan Boneh, Stanford University

Quick Recap

A **block cipher** is a pair of efficient algs. (E, D):

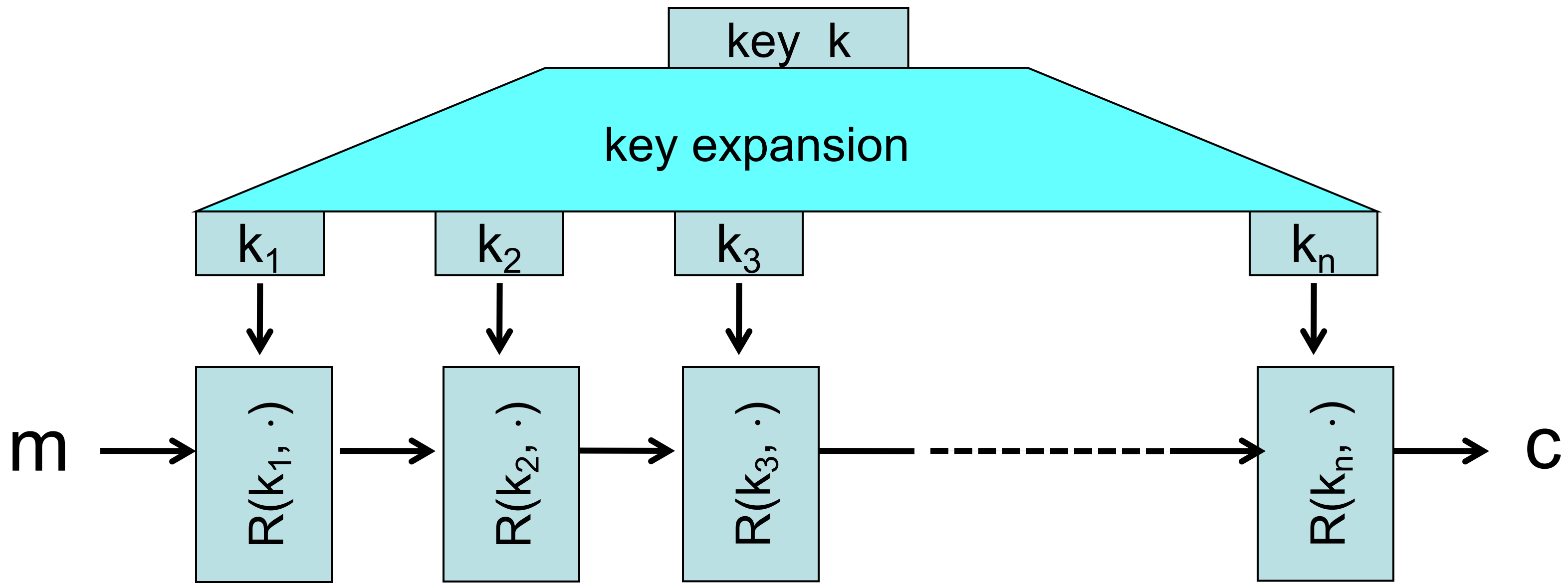


Canonical examples:

1. AES: n=128 bits, k = 128, 192, 256 bits

2. 3DES: n= 64 bits, k = 168 bits (historical)

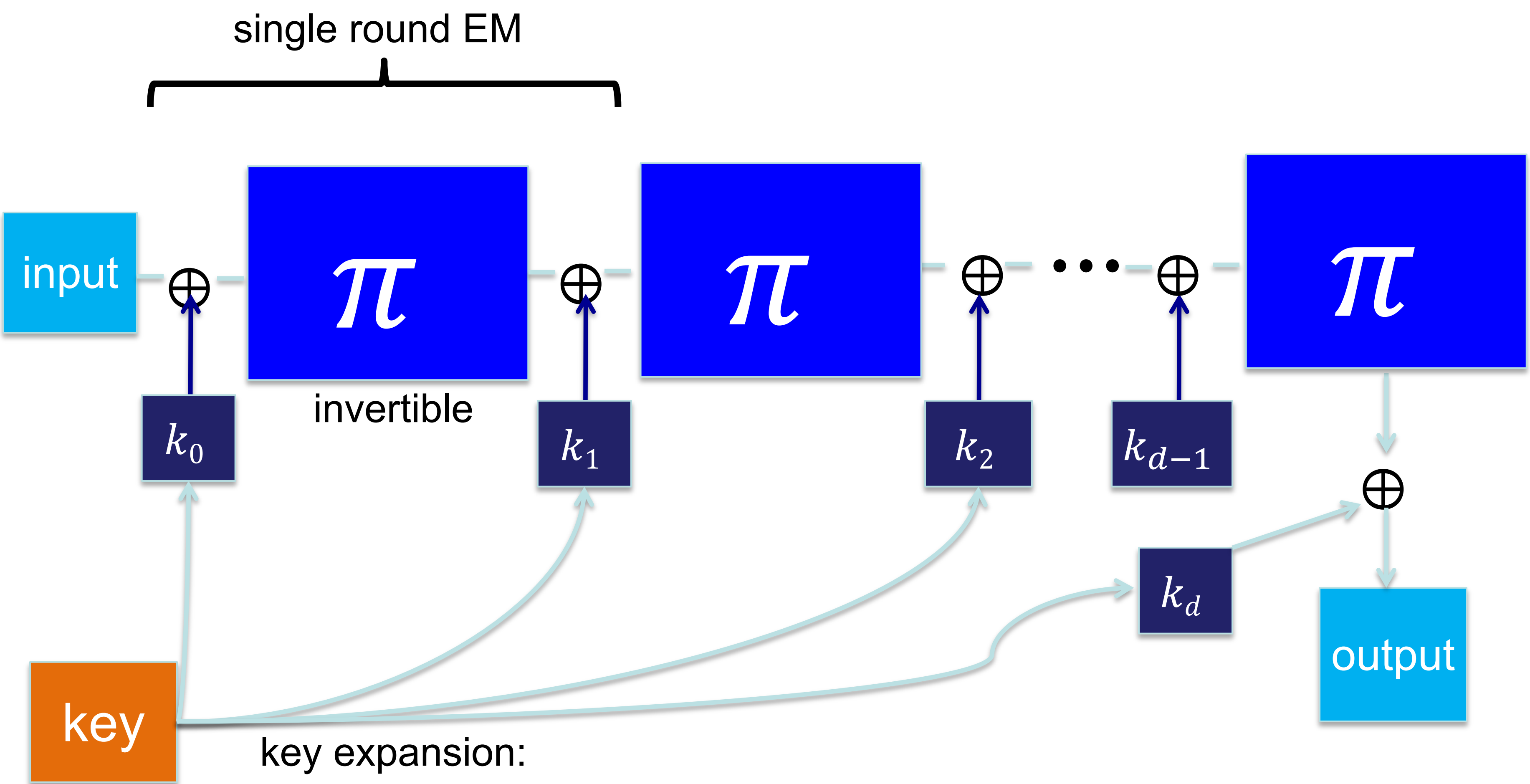
Block Ciphers Built by Iteration



R(k,m) is called a round function

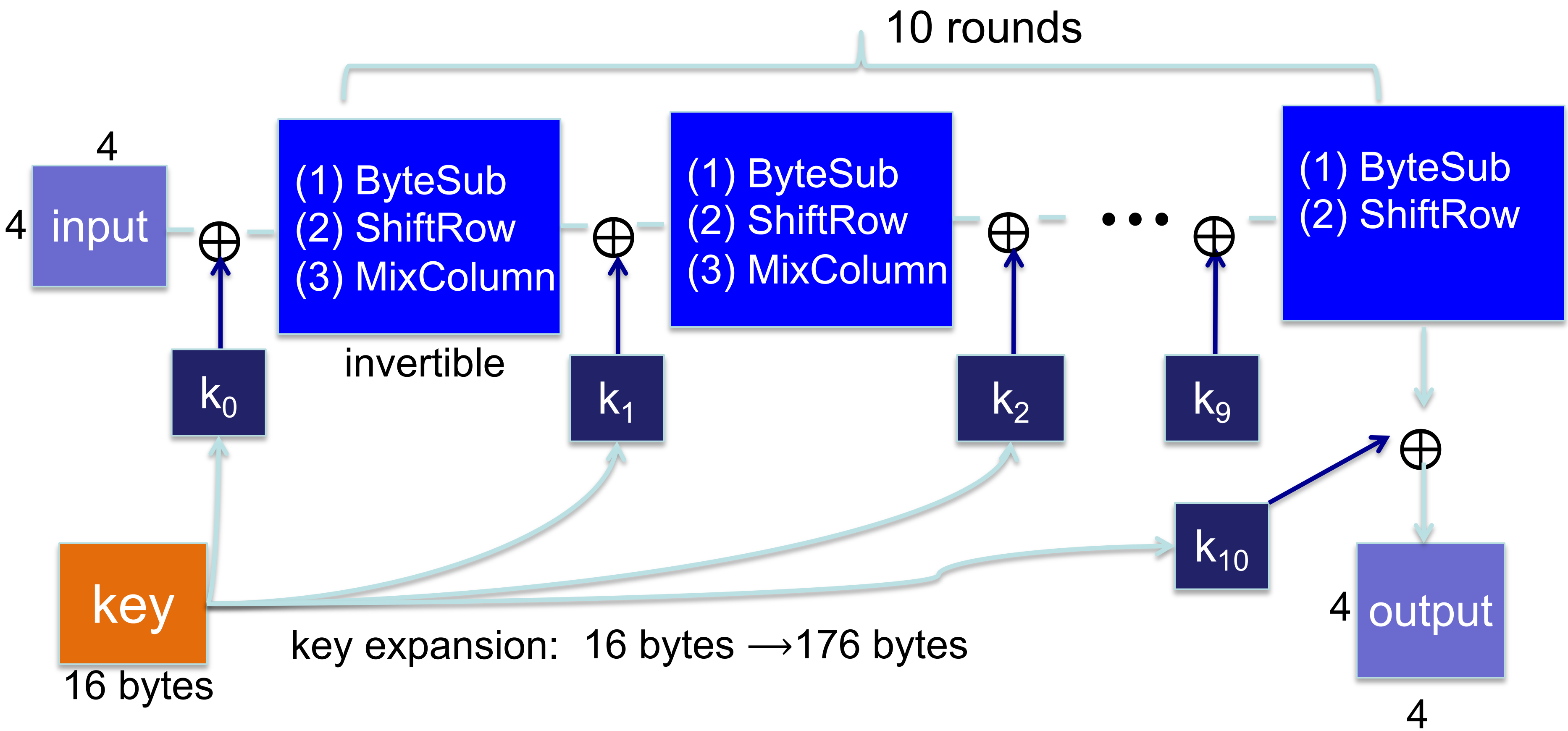
3DES: n=48, **AES128:** n=10, **AES256:** n=14

AES: an iterated Even-Mansour cipher



π: $\{0,1\}^n \rightarrow \{0,1\}^n$ invertible function

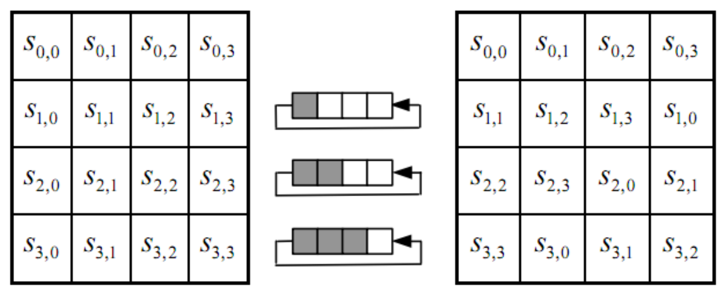
AES128: 10 rounds of EM



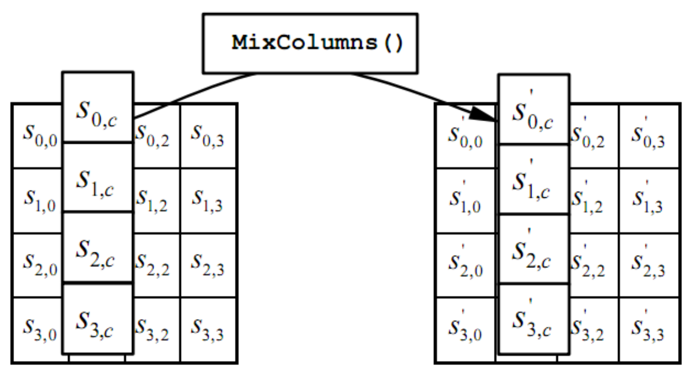
The permutation π

(1) ByteSub: a 1 byte S-box. 256 byte table. (invertible)

(2) ShiftRows:



(3) MixColumns:



Recall the AES pledge

I promise that I will not implement AES myself in production code, even though it might be fun. This agreement will remain in effect until I learn all about side-channel attacks and countermeasures to the point where I lose all interest in implementing AES myself.

Performance (no HW acceleration)

	Cipher	Block/key size	Speed (MB/sec)
	ChaCha20	- / 256	643
	3DES	64 / 168	30
block	AES128	128 / 128	163
	AES256	128 / 256	115

AES-NI: AES hardware instructions

AES instructions (Intel, AMD, ARM, …)

- **aesenc, aesenclast**: do one round of AES

  128-bit registers: xmm1=state, xmm2=round key

  aesenc xmm1, xmm2 ; puts result in xmm1

- **aesdec, aesdeclast:** one round of AES^{-1}
- **aeskeygenassist**: performs AES key expansion

Claim 1: 14 x speed-up over OpenSSL on same hardware

Claim 2: constant time execution

AES-NI: parallelism and pipelining

- Intel Skylake (old): 4 cycles for one aesenc
  - **fully pipelined**: can issue one instruction every cycle

- Intel Icelake (2019): **vectorized aesenc** (vaesenc)
  - **vaesenc**: compute aesenc on four blocks in parallel
  - fully pipelined

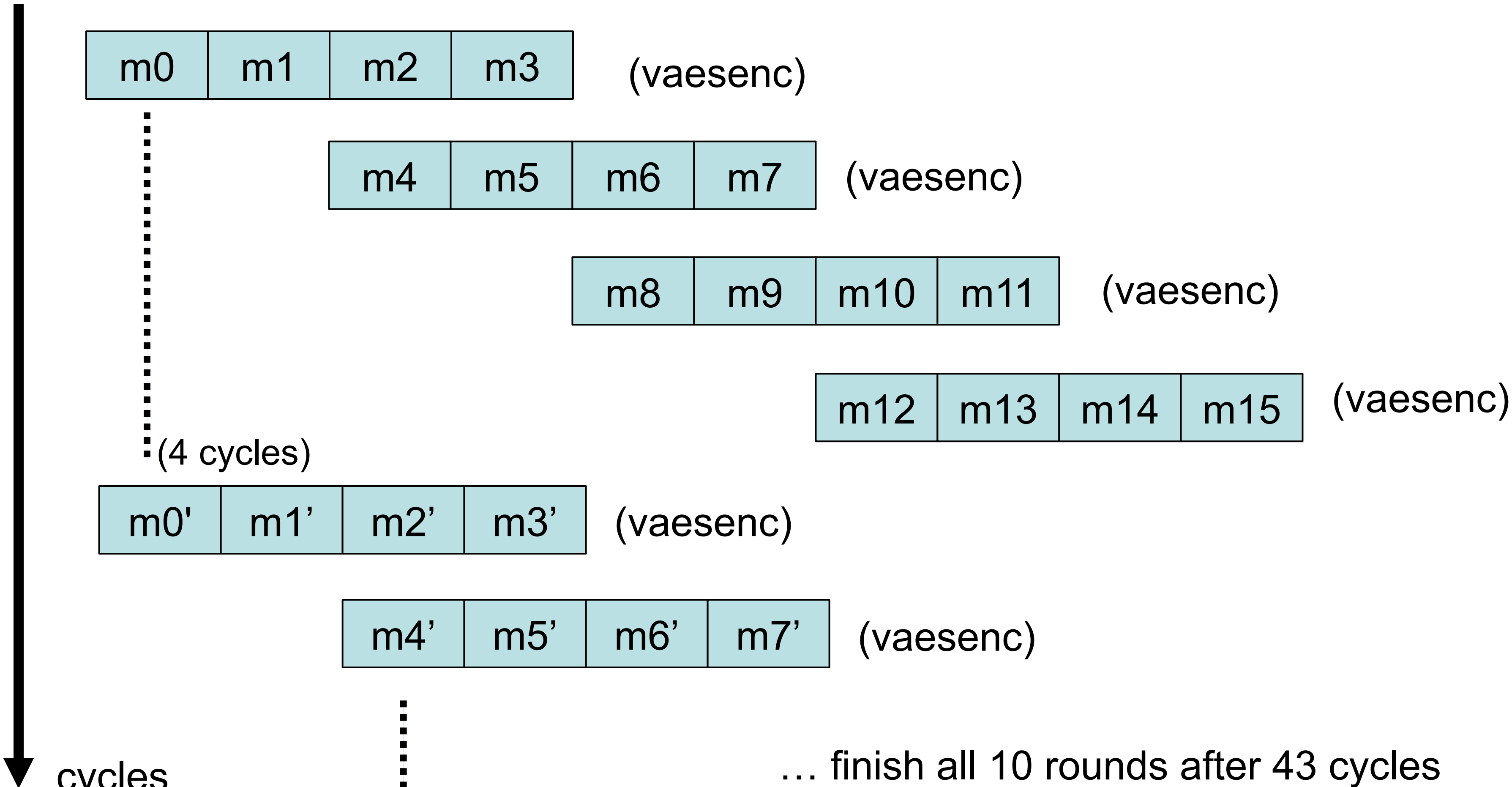
<u>**Implications:**</u>

- AES128 encrypt a <u>**single**</u> block takes **40 cycles** (10 rounds)

- AES128 encrypt 16 blocks on Icelake takes **43 cycles**

AES128 encrypt on Icelake

To encrypt 16 blocks do: m0, …, m15 $\in \{0,1\}^{128}$



PRPs and PRFs

Topics:

1. Abstract block ciphers: PRPs and PRFs
2. Security models for encryption
3. Analysis of CBC and counter mode

PRPs and PRFs

- Pseudo Random Function (**PRF**) defined over (K,X,Y):

$$F:\ K \times X \rightarrow Y$$

such that exists "efficient" algorithm to evaluate F(k,x)

- Pseudo Random Permutation (**PRP**) defined over (K,X):

$$E:\ K \times X \rightarrow X$$

such that:

1. Exists "efficient" algorithm to evaluate E(k,x)
2. The function $E(k, \cdot)$ is one-to-one
3. Exists "efficient" inversion algorithm D(k,x)

Running example

- <u>Example PRPs</u>: 3DES, AES, …

  AES128: $K \times X \rightarrow X$ where $K = X = \{0,1\}^{128}$

  DES: $K \times X \rightarrow X$ where $X = \{0,1\}^{64}$, $K = \{0,1\}^{56}$

  3DES: $K \times X \rightarrow X$ where $X = \{0,1\}^{64}$, $K = \{0,1\}^{168}$

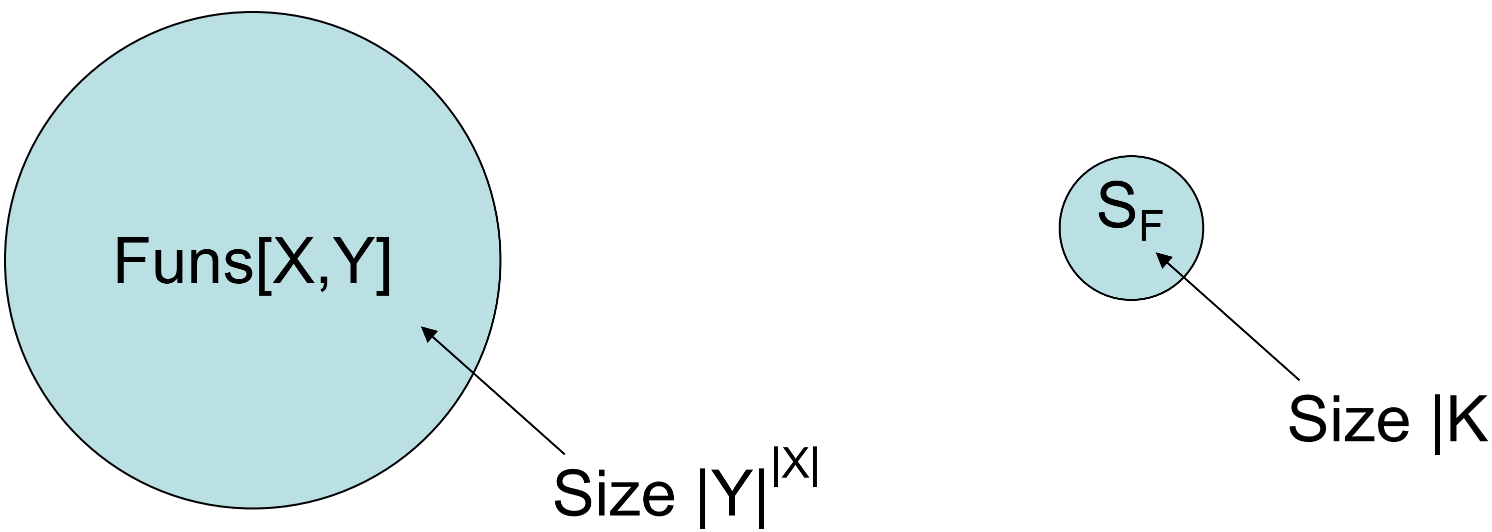
- Functionally, any PRP is also a PRF.
  - A PRP is a PRF where X=Y and is efficiently invertible
  - A PRP is sometimes called a ***block cipher***

Secure PRFs

- Let F: $K \times X \rightarrow Y$ be a PRF

$$\begin{cases} \text{Funs}[X,Y]: & \text{the set of } \underline{\textbf{all}} \text{ functions from } X \text{ to } Y \\ S_F = \{\ F(k,\cdot) \ \text{ s.t. } \ k \in K\ \} & \subseteq \quad \text{Funs}[X,Y] \end{cases}$$

- <u>Intuition</u>: a PRF is **secure** if
  a random function in Funs[X,Y] is indistinguishable from
  a random function in S_F



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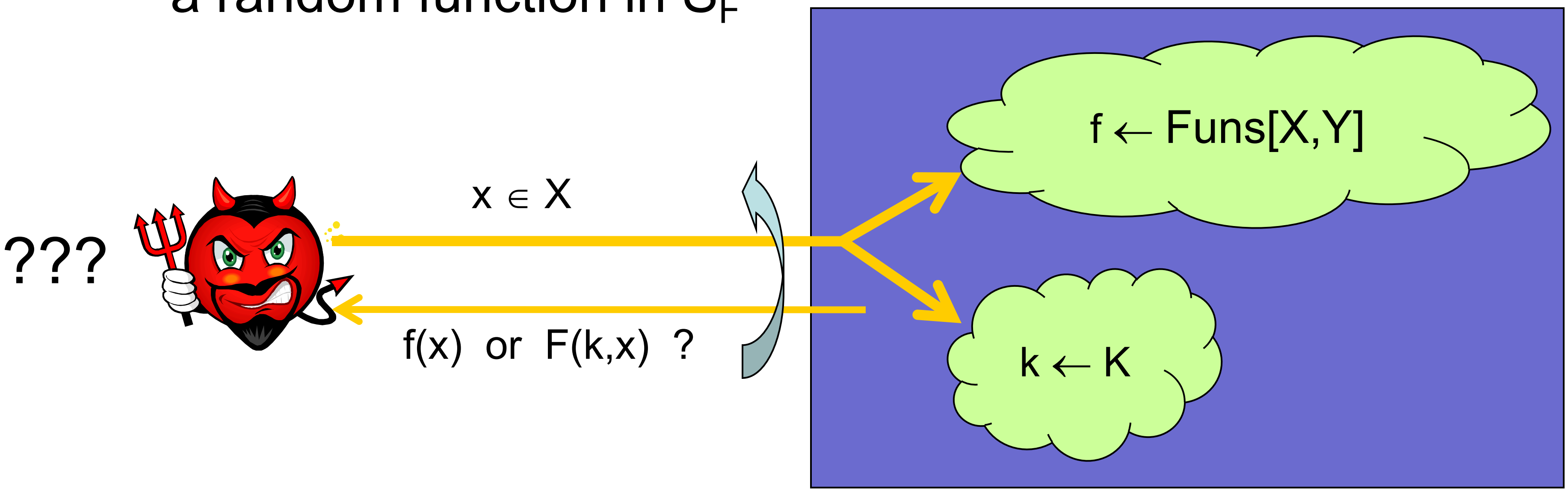
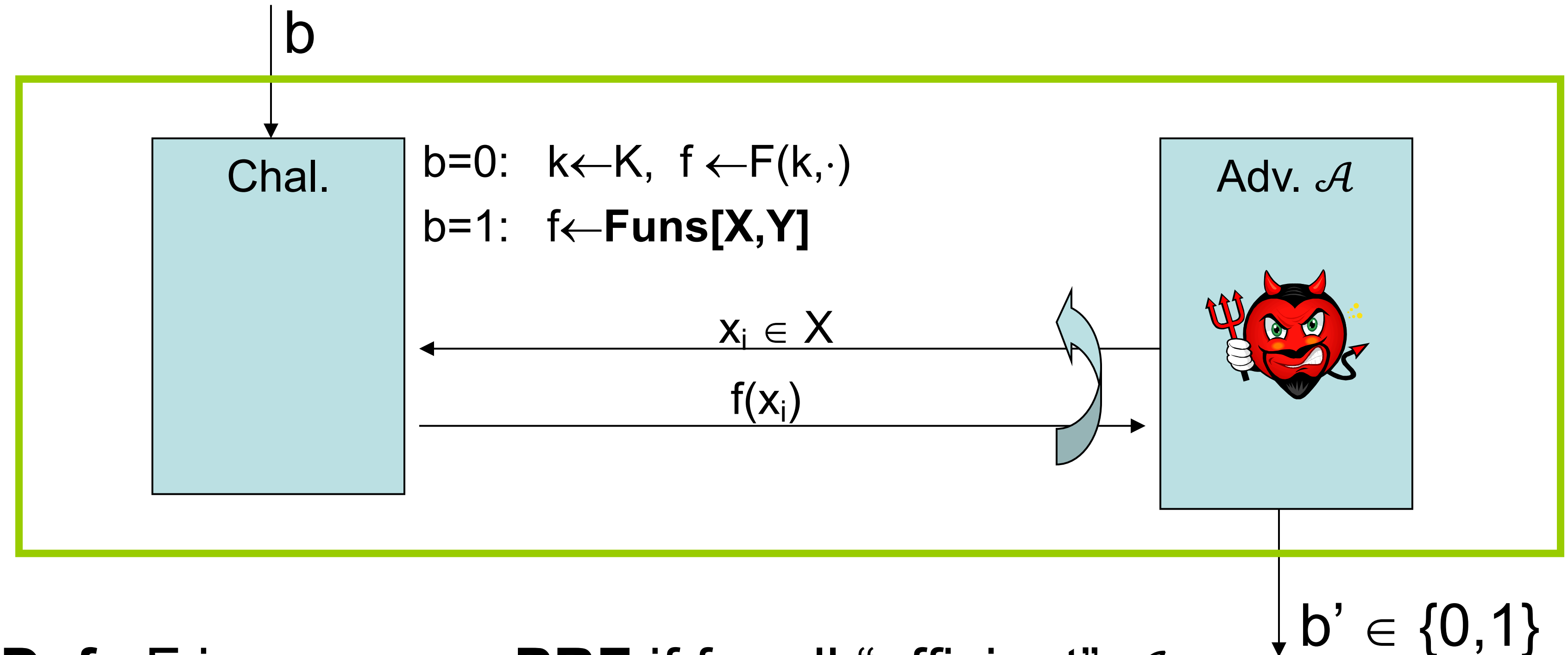


Secure PRF: defintion

- For b=0,1 define experiment EXP(b) as:



- **Def**: F is a **secure PRF** if for all "efficient" $\mathcal{A}$:

$$\mathrm{Adv}_{\mathrm{PRF}}[\mathcal{A},F] \;=\; \left|\Pr[\mathrm{EXP}(0) = 1] - \Pr[\mathrm{EXP}(1) = 1]\right|$$

is "negligible."

An example

Let $K = X = \{0,1\}^n$.

Consider the PRF: $\boxed{F(k, x) = k \oplus x}$ defined over (K, X, X)

Let's show that F is insecure:

Adversary $\mathcal{A}$: (1) choose arbitrary $x_0 \neq x_1 \in X$

(2) query for $y_0 = f(x_0)$ and $y_1 = f(x_1)$

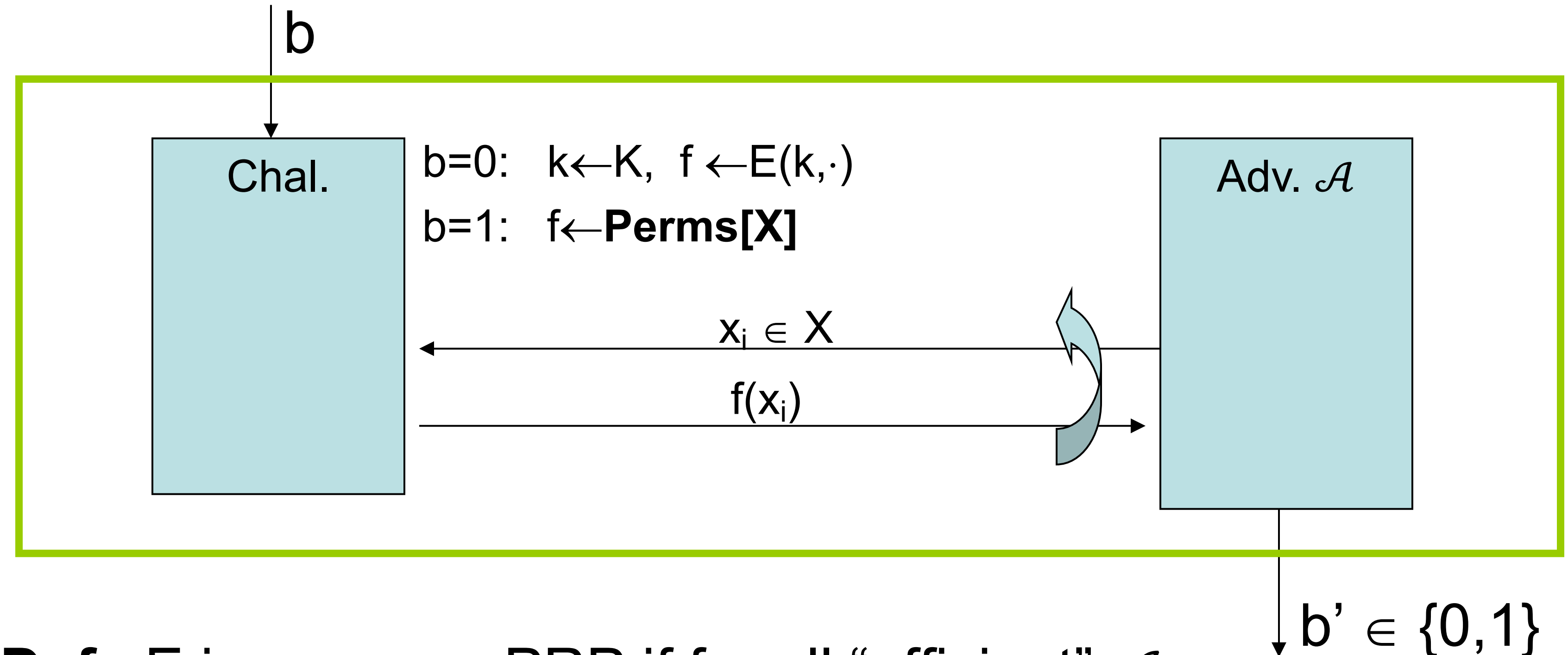
(3) output \`0' if $y_0 \oplus y_1 = x_0 \oplus x_1$ , else \`1'

$$\boxed{\Pr[\mathrm{EXP}(0) = 0] = 1} \qquad \boxed{\Pr[\mathrm{EXP}(1) = 0] = 1/2^n}$$

$$\Longrightarrow \quad \mathrm{Adv}_{\mathrm{PRF}}[\mathcal{A},F] = 1 - (1/2^n) \qquad \text{(not negligible)}$$

Secure PRP

- For b=0,1 define experiment EXP(b) as:



- **Def**: E is a secure PRP if for all "efficient" $\mathcal{A}$:

$$\mathrm{Adv}_{\mathrm{PRP}}[\mathcal{A},E] = \left|\Pr[\mathrm{EXP}(0) = 1] - \Pr[\mathrm{EXP}(1) = 1]\right|$$

is "negligible."

Example secure PRPs

- Example secure PRPs: 3DES, AES, …

  AES256: $K \times X \to X$ where $X = \{0,1\}^{128}$

  $K = \{0,1\}^{256}$

- AES256 PRP Assumption (example) :

  For all $\mathcal{A}$ s.t. time($\mathcal{A}$) < 2^{80} : $\mathrm{Adv}_{\mathrm{PRP}}[\mathcal{A}, \textbf{AES256}] < 2^{-40}$

The PRP-PRF Switching Lemma

Any secure PRP is also a secure PRF.

<u>Lemma</u>: Let E be a PRP over (K, X).

Then for any q-query adversary $\mathcal{A}$:

$$\left| \text{Adv}_{\text{PRF}}[\mathcal{A},\text{E}] - \text{Adv}_{\text{PRP}}[\mathcal{A},\text{E}] \right| < q^2 / 2|X|$$

$\Rightarrow$ Suppose |X| is large so that $q^2 / 2|X|$ is "negligible"

Then $\text{Adv}_{\text{PRP}}[\mathcal{A},\text{E}]$ "negligible" $\Rightarrow$ $\text{Adv}_{\text{PRF}}[\mathcal{A},\text{E}]$ "negligible"

Using PRPs and PRFs

- Goal: build "secure" encryption from a PRP.

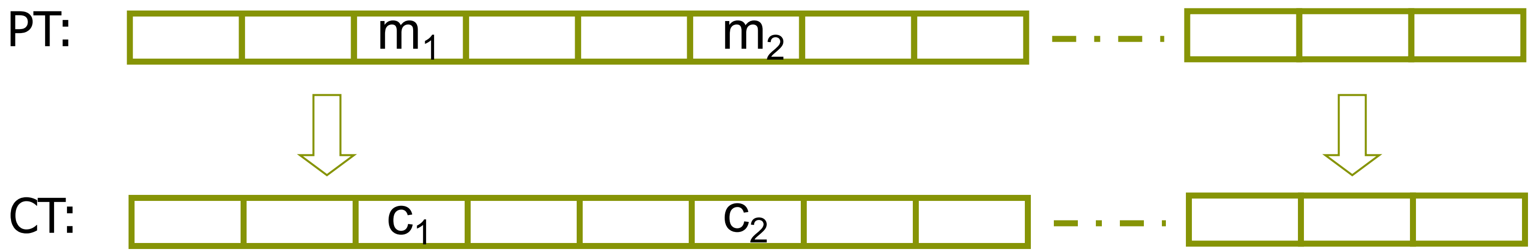
- Security is always defined using two parameters:

  1. What "**power**" does adversary have?
     examples:
     - Adv sees only one ciphertext (one-time key)
     - Adv sees many PT/CT pairs (many-time key, CPA)

  2. What "**goal**" is adversary trying to achieve?
     examples:
     - Fully decrypt a challenge ciphertext.
     - Learn info about PT from CT (semantic security)

Incorrect use of a PRP

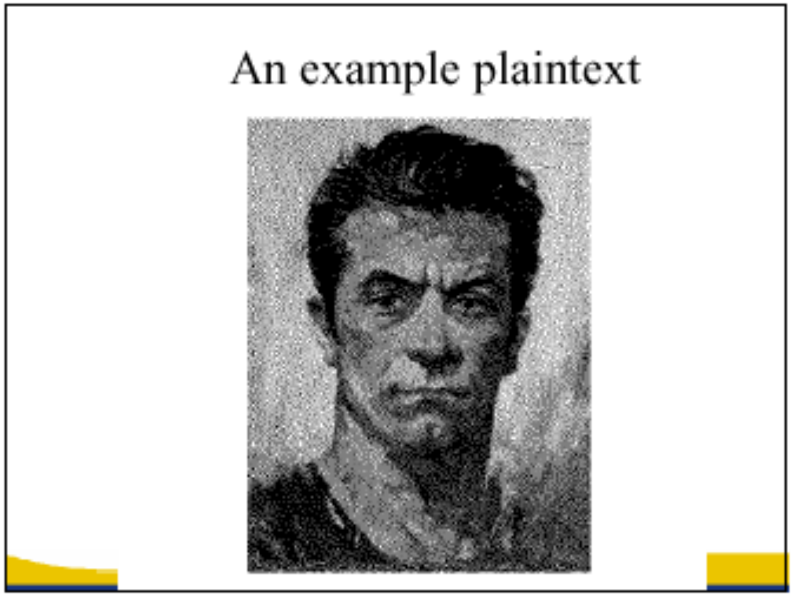
Electronic Code Book (ECB):

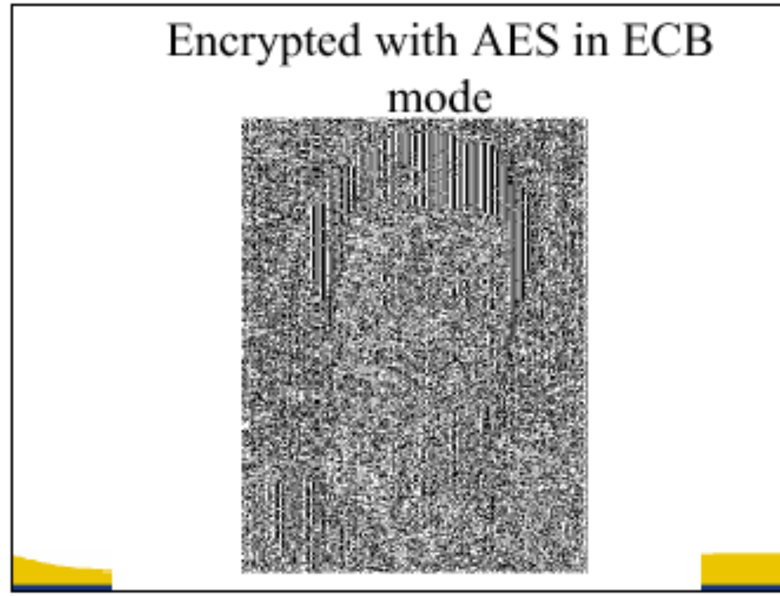


Problem:

– if $m_1=m_2$ then $c_1=c_2$

In pictures





(courtesy B. Preneel)

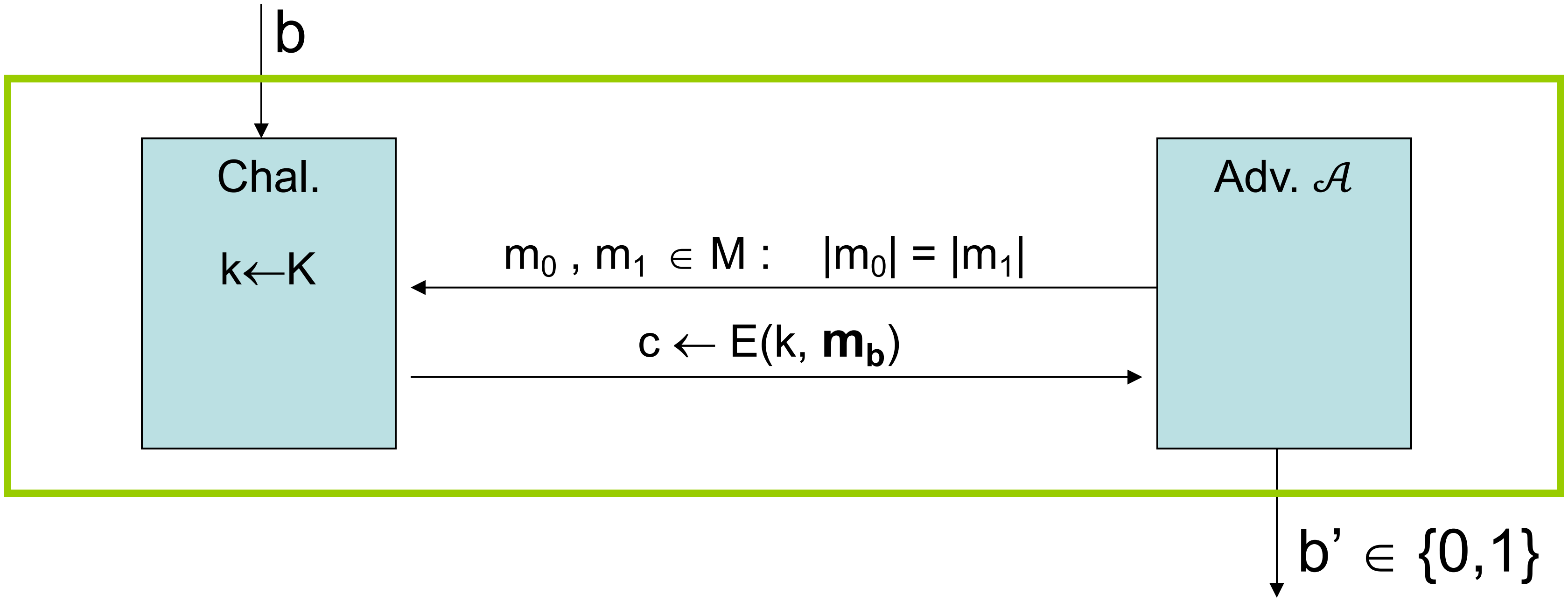
Modes of Operation for One-time Use Key

Example application:

Encrypted email. New key for every message.

Semantic Security for one-time key

- $\mathbb{E}$ = (E,D) a cipher defined over (K,M,C)
- For b=0,1 define EXP(b) as:



- Def: $\mathbb{E}$ is sem. sec. for one-time key if for all "efficient" $\mathcal{A}$:

$$\text{Adv}_{\text{SS}}[\mathcal{A},\mathbb{E}] = \left| \Pr[\text{EXP}(0)=1] - \Pr[\text{EXP}(1)=1] \right|$$

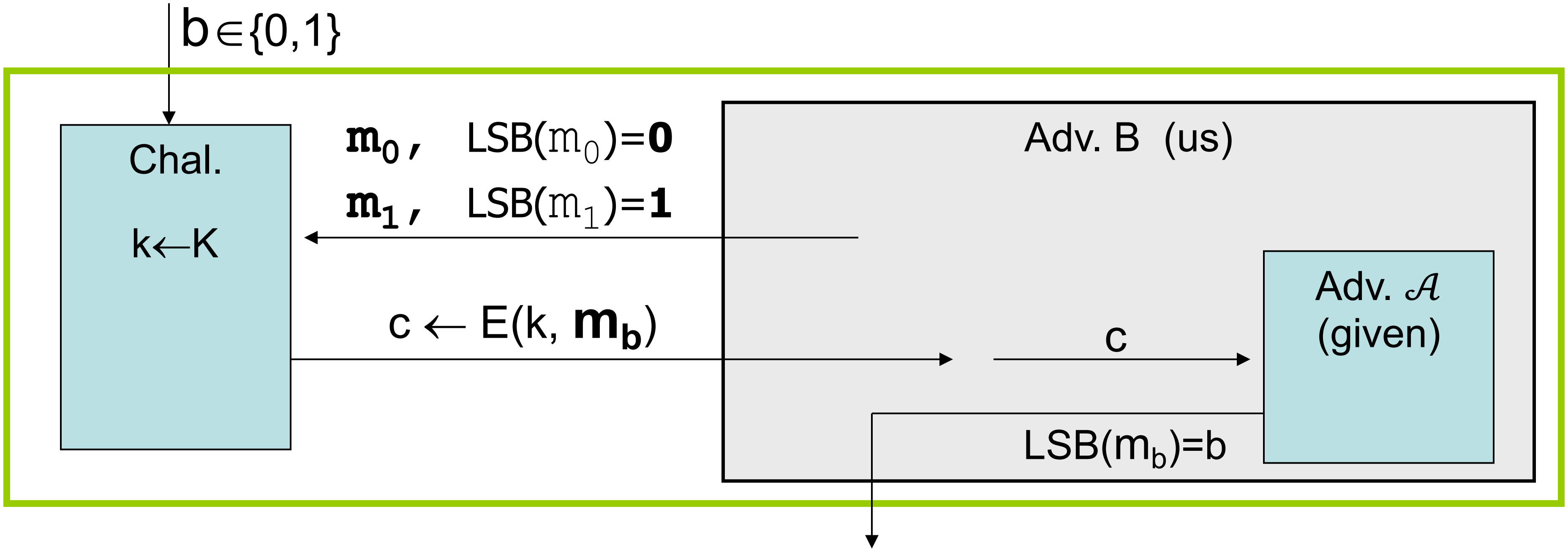
is "negligible."

Semantic security (cont.)

Sem. Sec. $\Rightarrow$ no "efficient" adversary learns "info" about PT from a **single** CT.

Example: suppose efficient $\mathcal{A}$ can deduce LSB of PT from CT.

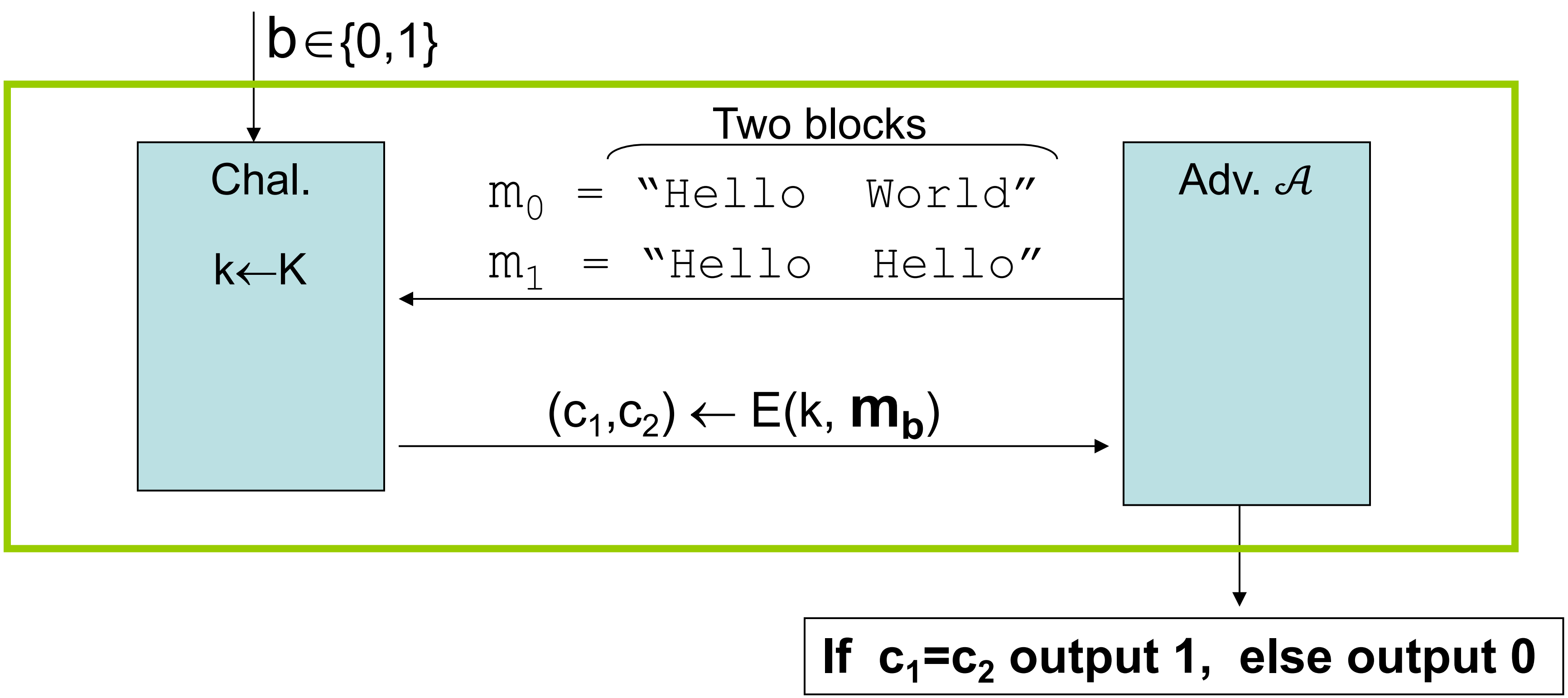
Then $\mathbb{E}$ = (E,D) is not semantically secure.



Then $\text{Adv}_{SS}[B, \mathbb{E}] = 1$ $\Rightarrow$ $\mathbb{E}$ is not sem. sec.

Note: ECB is not Sem. Sec.

ECB is not semantically secure for messages that contain two or more blocks.



Then $\text{Adv}_{\text{SS}}[\mathcal{A}, \text{ECB}] = 1$

Secure Constructions

Examples of sem. sec. systems:

1. $Adv_{SS}[\mathcal{A}, OTP] = 0$ for **<u>all</u>** $\mathcal{A}$

2. Deterministic counter mode from a PRF F :
   - $E_{DETCTR}(k,m)$ =

$$
\begin{array}{c|c|c|c|c|}
 & m[0] & m[1] & \dots & m[L] \\
\oplus & & & & \\
 & \mathbf{F(k,0)} & \mathbf{F(k,1)} & \dots & \mathbf{F(k,L)} \\
\hline
 & c[0] & c[1] & \dots & c[L] \\
\end{array}
$$

   - Stream cipher built from PRF (e.g. AES)

Det. counter-mode security

Theorem: For any L>0.

If F is a secure PRF over (K,X,X) then

E_{DETCTR} is sem. sec. cipher over (K, X^L, X^L).

In particular, for any adversary $\mathcal{A}$ attacking E_{DETCTR} there exists a PRF adversary B s.t.:

$$\mathrm{Adv}_{SS}[\mathcal{A}, E_{DETCTR}] = 2 \cdot \mathrm{Adv}_{PRF}[B, F]$$

$\mathrm{Adv}_{PRF}[B, F]$ is negligible (since F is a secure PRF)

$\Rightarrow$ $\mathrm{Adv}_{SS}[\mathcal{A}, E_{DETCTR}]$ must be negligible.

Modes of Operation for Many-time Key

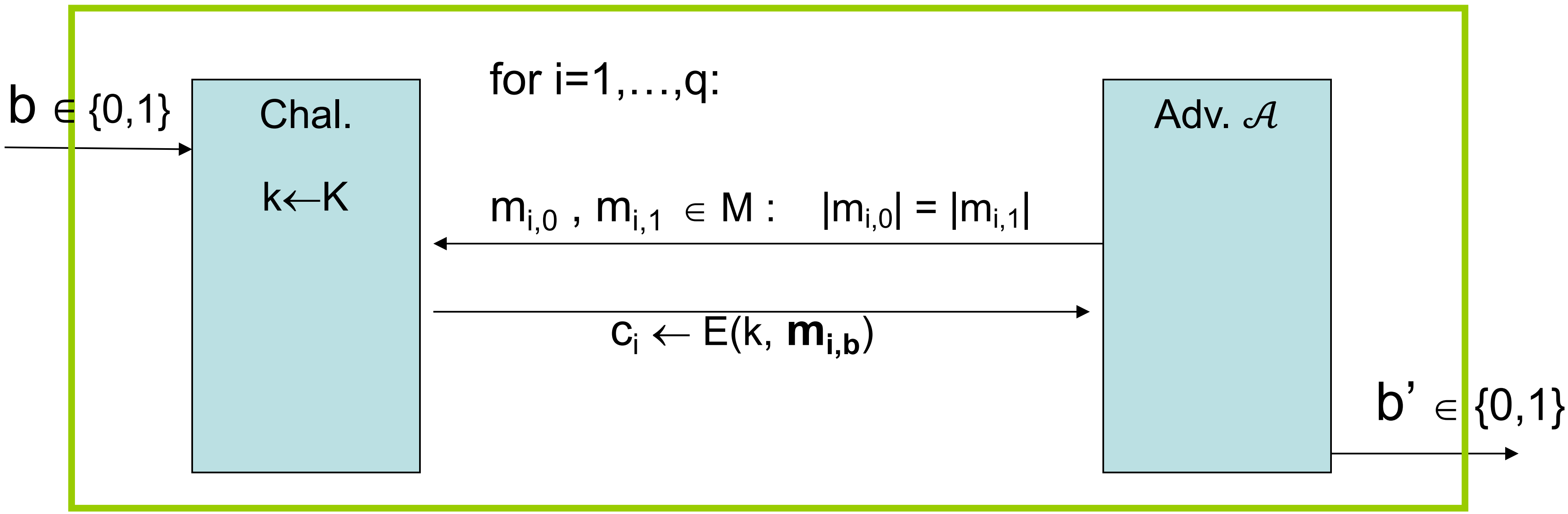
Example applications:

1. File systems: Same AES key used to encrypt many files.
2. IPsec: Same AES key used to encrypt many packets.

Semantic Security for many-time key (CPA security)

Cipher $\mathbb{E} = (E,D)$ defined over (K,M,C).

For $b=0,1$ define EXP(b) as:



if adv. wants $c = E(k, m)$ it queries with $m_{j,0} = m_{j,1} = m$

Def: $\mathbb{E}$ is sem. sec. under CPA if for all "efficient" $\mathcal{A}$:

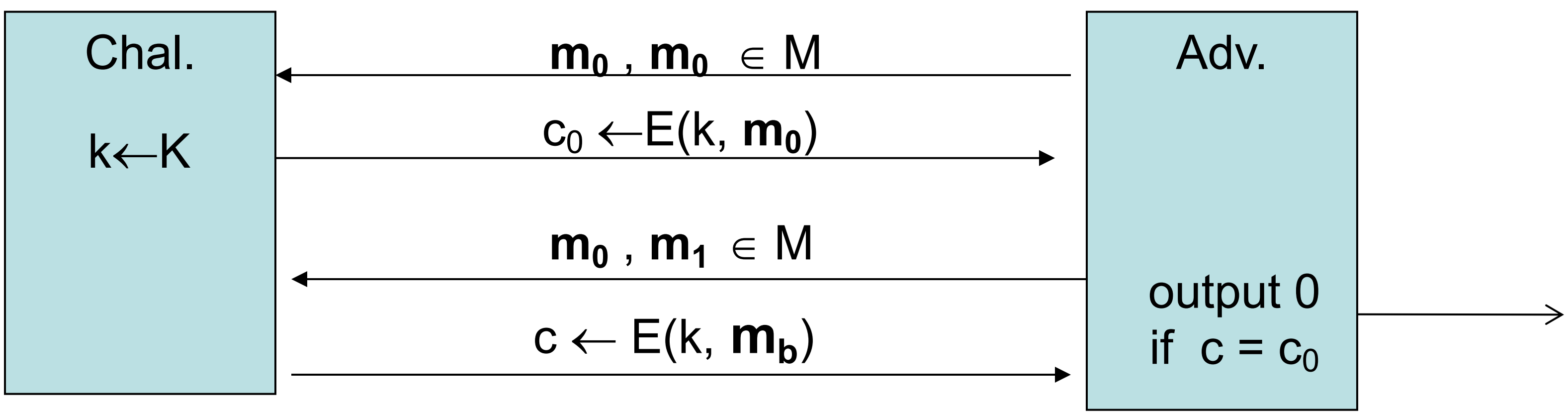
$$\mathrm{Adv}_{\mathrm{CPA}}[\mathcal{A},\mathbb{E}] = \Big|\Pr[\mathrm{EXP}(0)=1] - \Pr[\mathrm{EXP}(1)=1]\Big|$$

is "negligible."

Security for many-time key

<u>Fact:</u> stream ciphers are insecure under CPA.

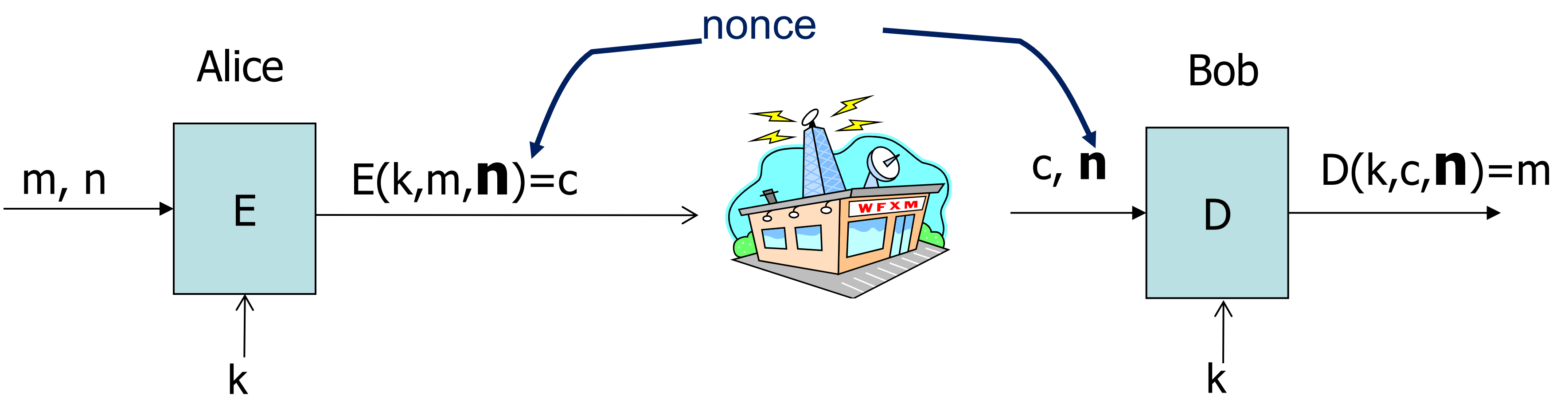
– More generally: if E(k,m) always produces same ciphertext, then cipher is insecure under CPA.



If secret key is to be used multiple times $\Rightarrow$

given the same plaintext message twice,
the encryption alg. must produce different outputs.

Nonce-based Encryption



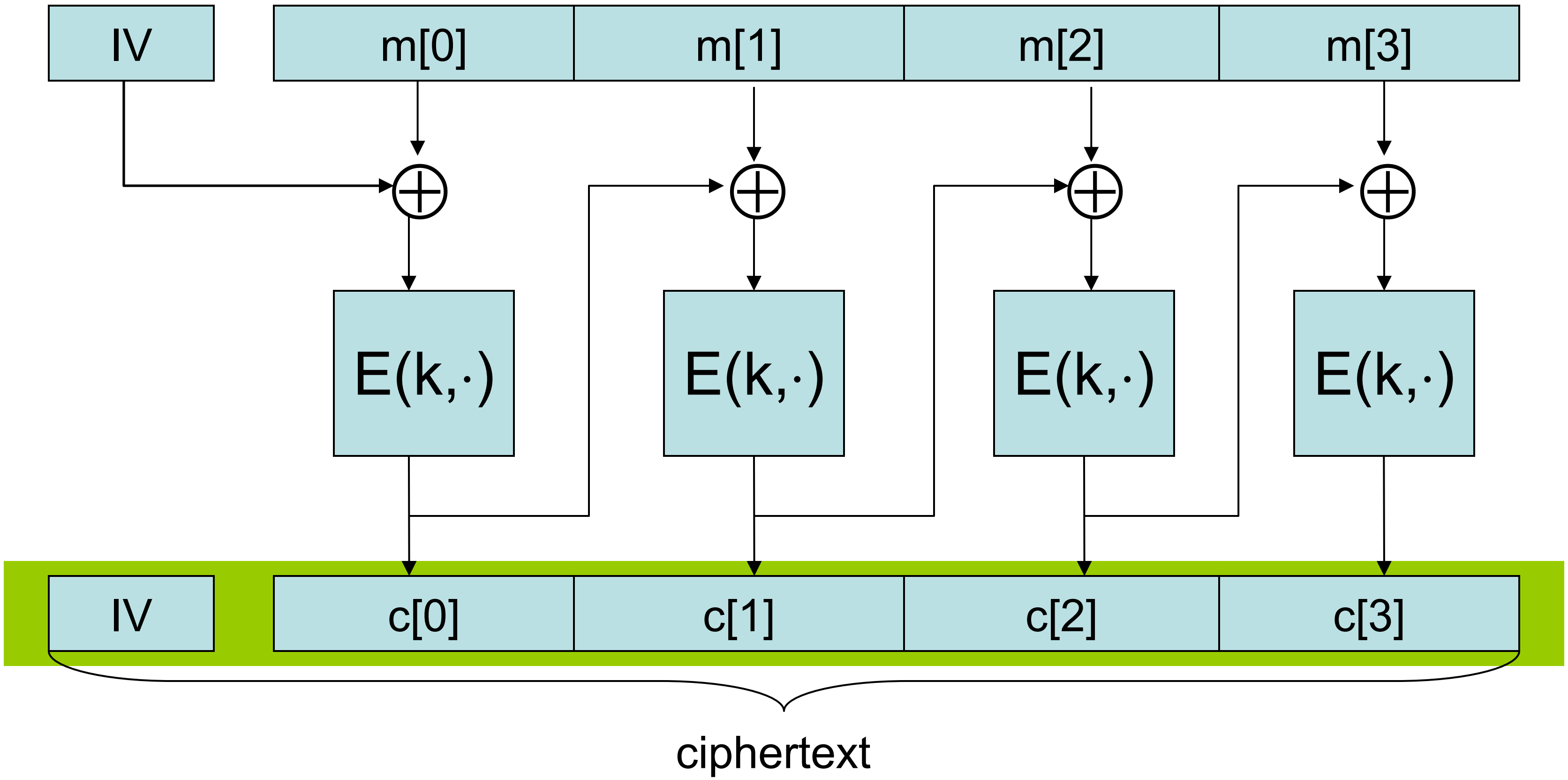
nonce n: a value that changes from msg to msg

(k,n) pair <u>never</u> used more than once

- <u>method 1</u>: encryptor chooses a random nonce, $n \leftarrow \mathcal{N}$
- <u>method 2</u>: nonce is a counter (e.g. packet counter)
  - used when encryptor keeps state from msg to msg
  - if decryptor has same state, need not send nonce with CT

Construction 1: CBC with random nonce

Cipher block chaining with a <u>random</u> IV (IV = nonce)



note: CBC where attacker can predict the IV is not CPA-secure. HW.

CBC: CPA Analysis

<u>CBC Theorem</u>: For any L>0,

If E is a secure PRP over (K,X) then

E_{CBC} is a sem. sec. under CPA over (K, X^L, X^{L+1}).

In particular, for a q-query adversary A attacking E_{CBC} there exists a PRP adversary B s.t.:

$$\mathrm{Adv}_{CPA}[A, E_{CBC}] \le 2 \cdot \mathrm{Adv}_{PRP}[B, E] + 2\, q^2\, L^2 / |X|$$

Note: CBC is only secure as long as $q^2 \cdot L^2 \ll |X|$

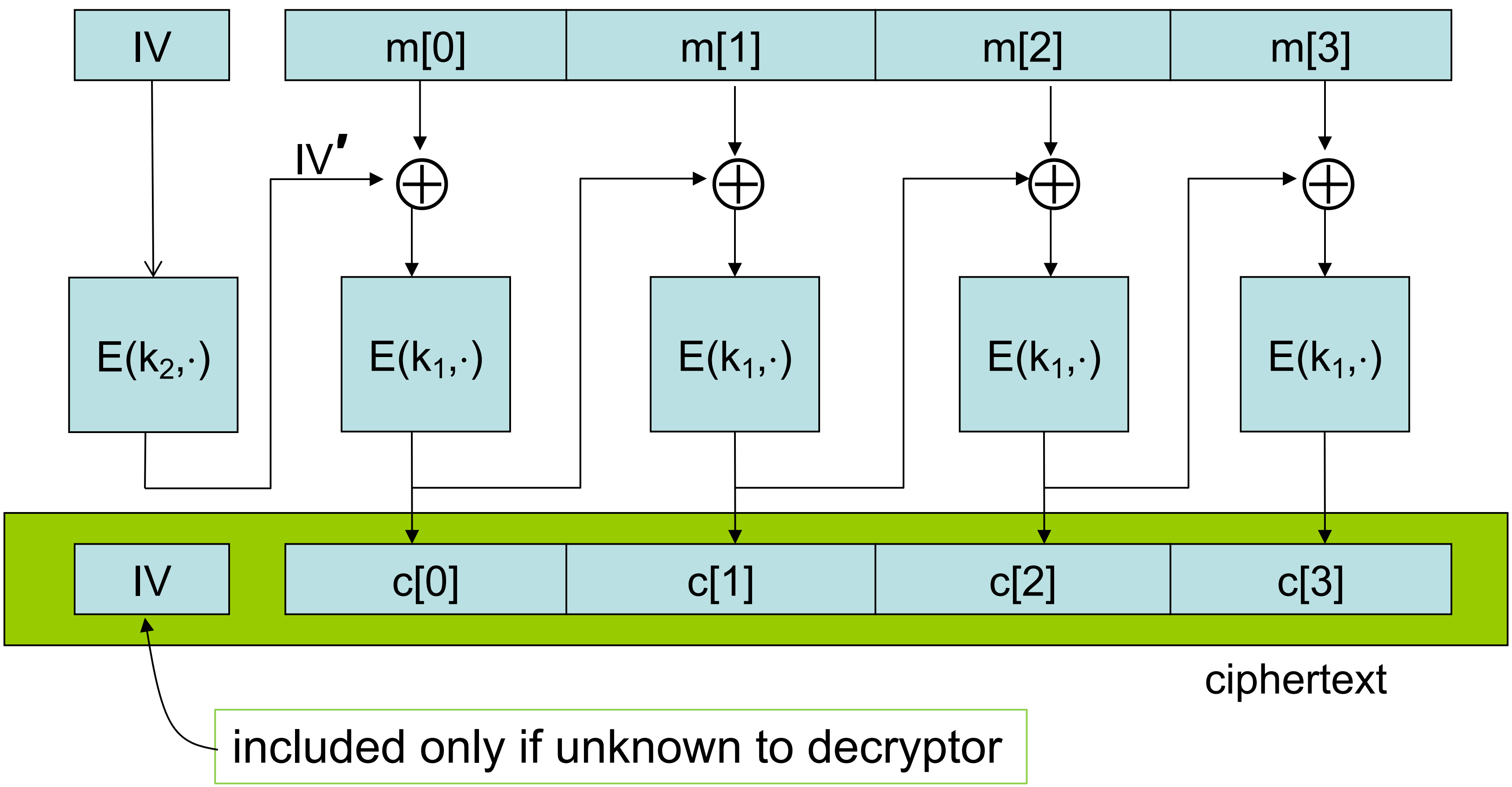
messages enc. with key

max msg length

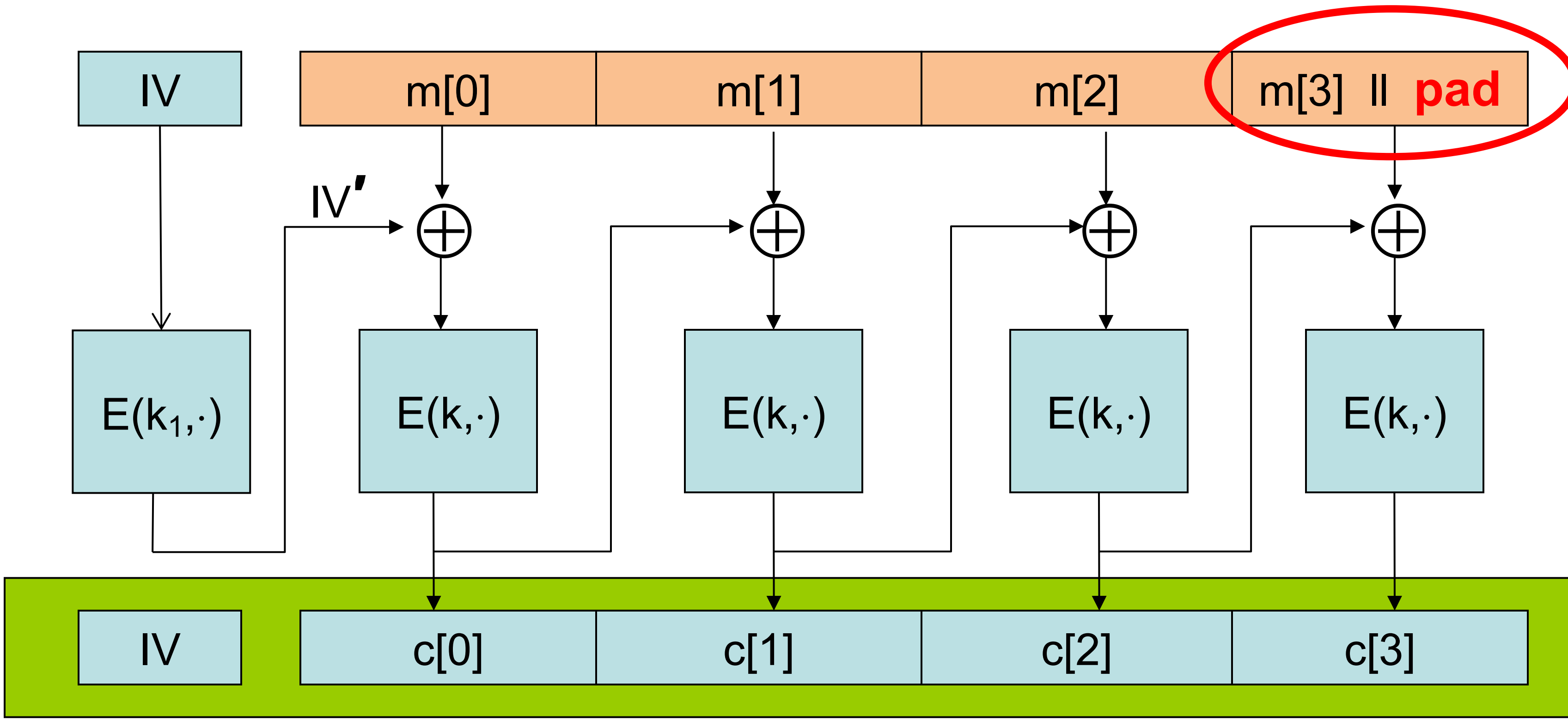
Construction 1': CBC with **unique** nonce

Cipher block chaining with unique IV (IV = nonce)

unique IV means: (key,IV) pair is used for only one message



A CBC technicality: padding



TLS 1.0: if need n-byte pad, n>0, use:

n-1	n-1	• • •	n-1

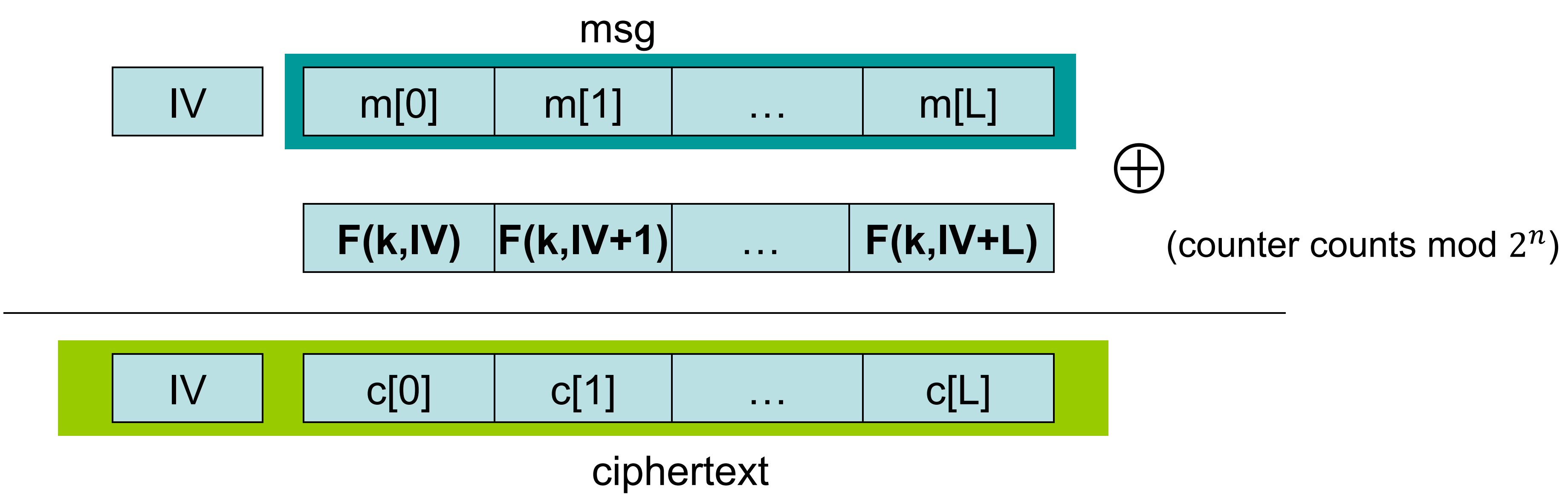
if no pad needed, add a dummy block

pad is removed during decryption

Construction 2: rand ctr-mode

F: PRF defined over (K,X,Y) where X = $\{0,1,\ldots,2^n\text{-}1\}$ and Y = $\{0,1\}^n$

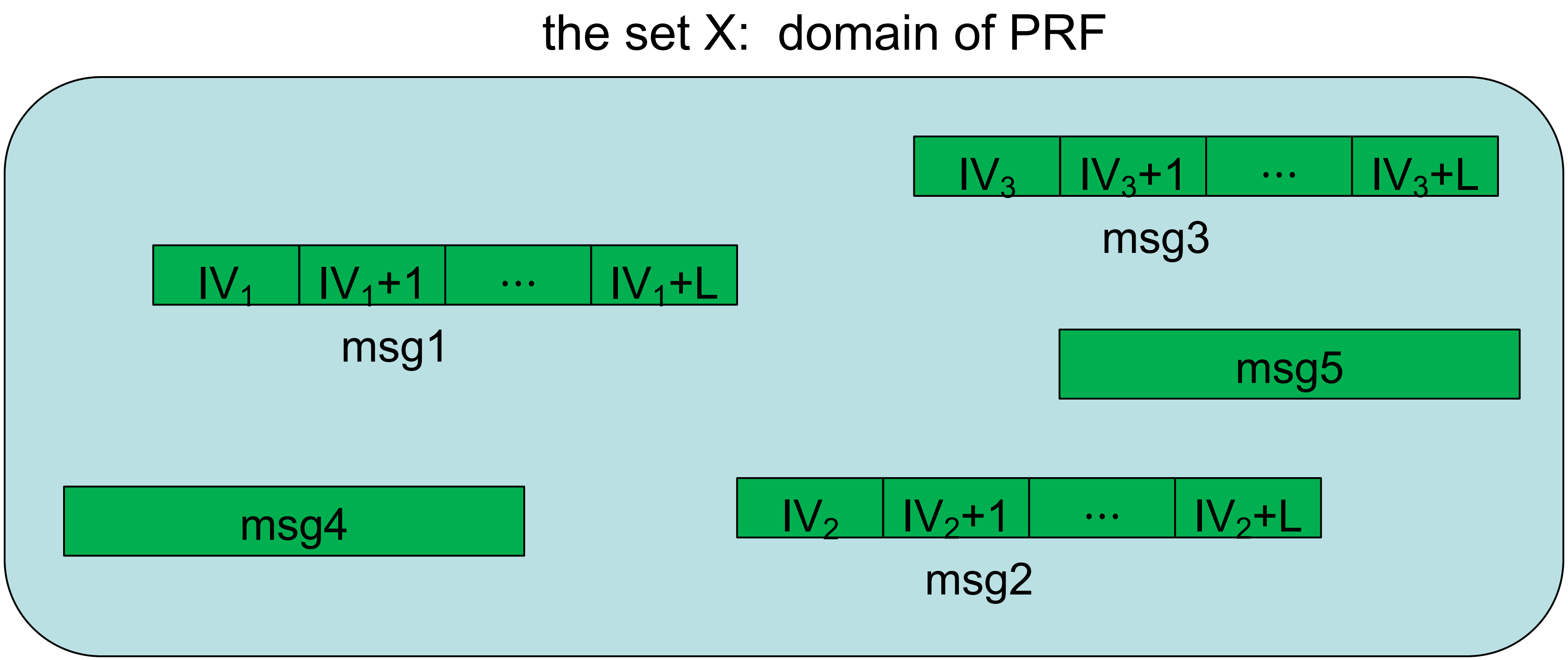
(e.g., n=128)



IV - chosen at random for every message

note: parallelizable (unlike CBC)

Why is this CPA secure?



CPA security holds as long as intervals do not intersect

- q msgs, L blocks each $\Rightarrow$ Pr[intersection] ≤ $\underbrace{2\, q^2\, L / |X|}_{\text{needs to be negligible}}$

rand ctr-mode: CPA analysis

Randomized counter mode: random IV.

<u>Counter-mode Theorem</u>: For any L>0,

If F is a secure PRF over (K,X,X) then

E_{CTR} is a sem. sec. under CPA over (K,X^L,X^{L+1}).

In particular, for a q-query adversary A attacking E_{CTR} there exists a PRF adversary B s.t.:

$$\mathrm{Adv}_{\mathrm{CPA}}[A, E_{\mathrm{CTR}}] \leq 2\cdot\mathrm{Adv}_{\mathrm{PRF}}[B, F] + 2\, q^2 L / |X|$$

<u>Note</u>: ctr-mode only secure as long as $q^2 \cdot L \ll |X|$

Better then CBC !

An example

$$\mathrm{Adv}_{\mathrm{CPA}}[A, E_{\mathrm{CTR}}] \leq 2\cdot\mathrm{Adv}_{\mathrm{PRF}}[B, E] + \mathbf{2\, q^2 L / |X|}$$

q = # messages encrypted with k , L = length of max msg

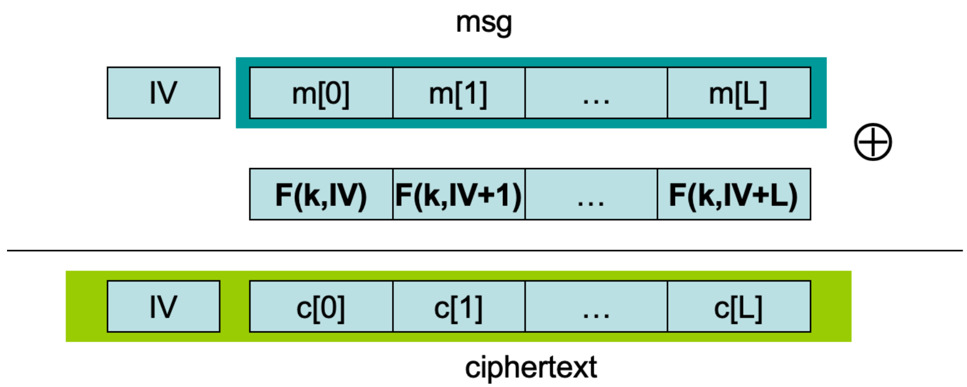
Suppose we want $\mathbf{Adv_{CPA}}[A, E_{\mathrm{CTR}}] \leq \mathbf{1/ 2^{31}}$

- Then need: $\mathbf{q^2 L / |X|} \leq \mathbf{1/ 2^{32}}$
- AES: $|X| = 2^{128} \Rightarrow \mathbf{q}\ \mathbf{L}^{1/2} < 2^{48}$

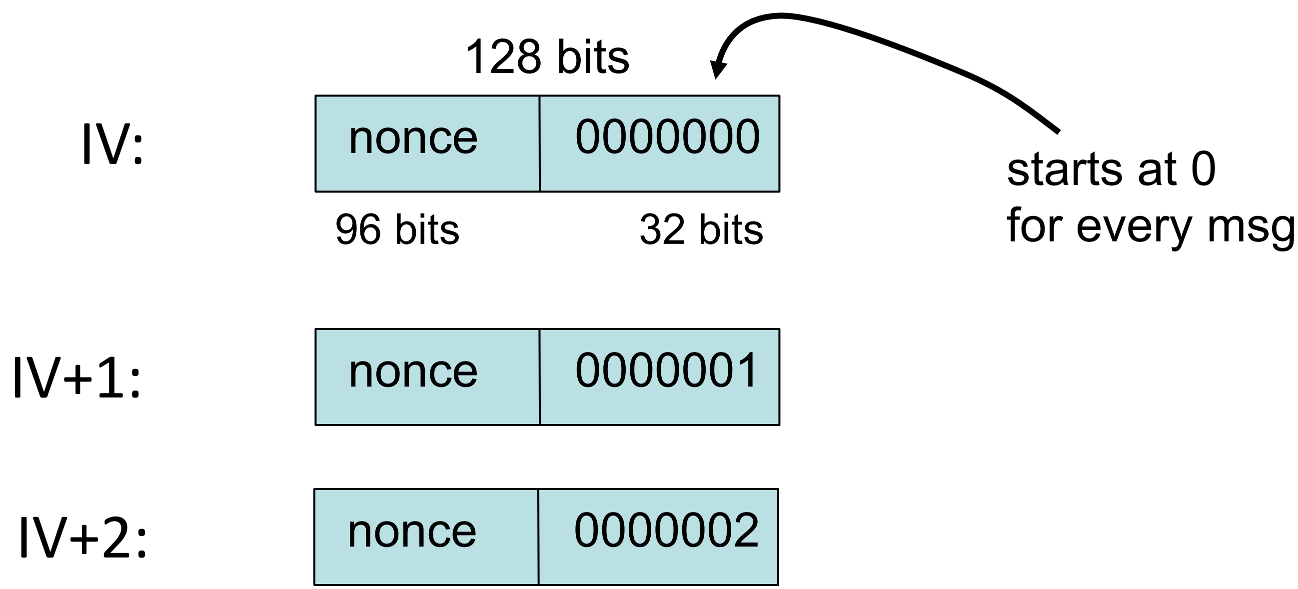
So, after **2^{32} CTs** each of **len 2^{32}** , must change key

(total of 2^{64} AES blocks)

Construction 2’: nonce ctr-mode



To ensure F(k,x) is never used more than once, choose IV as:



Comparison: ctr vs. CBC

	~~**CBC**~~	**ctr mode**
required primitive	~~PRP~~	PRF
parallel processing	~~No~~	Yes
security	~~q^2 L^2 << |X|~~	q^2 L << |X|
dummy padding block	~~Yes*~~	No
1 byte msgs (nonce-based)	~~16x expansion~~	no expansion

* for CBC, dummy padding block can be avoided using *ciphertext stealing*

Summary

PRPs and PRFs: a useful abstraction of block ciphers.

We examined two security notions:

1. Semantic security against one-time.
2. Semantic security against many-time CPA.

Note: neither mode ensures data integrity.

Stated security results summarized in the following table:

Power / Goal	**one-time key**	**Many-time key (CPA)**	**CPA and CT integrity**
Sem. Sec.	steam-ciphers det. ctr-mode	rand CBC rand ctr-mode	later

Attacks on block ciphers

Goal: distinguish block cipher from a random permutation

- if this can be done efficiently then block cipher is broken

Harder goal:

find key k given many $c_i = E(k, m_i)$ for random m_i

(1) Linear and differential attacks

[BS'89,M'93]

Given *many* (m_i, c_i) pairs, can recover key much faster than exhaustive search

<u>Linear cryptanalysis</u> (overview) : let c = DES(k, m)

Suppose for random k, m :

$$\Pr\Big[\; m[i_1]\oplus\cdots\oplus m[i_r] \;\bigoplus\; c[j_j]\oplus\cdots\oplus c[j_v] \;=\; k[l_1]\oplus\cdots\oplus k[l_u] \;\Big] = \tfrac{1}{2} + \varepsilon$$

For some ε.

For DES, this exists with $\varepsilon = 1/2^{21} \approx 0.0000000477$!!

Linear attacks

$$\Pr\Big[\, m[i_1]\oplus\cdots\oplus m[i_r] \;\bigoplus\; c[j_j]\oplus\cdots\oplus c[j_v] \;=\; k[l_1]\oplus\cdots\oplus k[l_u] \,\Big] = ½ + \varepsilon$$

<u>Thm</u>: given $1/\varepsilon^2$ random pairs $\big(m,\ c=DES(k, m)\big)$ then

$$k[l_1]\oplus\ldots\oplus k[l_u] \;=\; \mathrm{MAJ}\Big[\, m[i_1]\oplus\ldots\oplus m[i_r] \;\bigoplus\; c[j_j]\oplus\ldots\oplus c[j_v] \,\Big]$$

with prob. ≥ 97.7%

⇒ with $1/\varepsilon^2$ inp/out pairs can find $k[l_1]\oplus\ldots\oplus k[l_u]$ in time $\approx 1/\varepsilon^2$

Linear attacks

For DES, $\varepsilon = 1/2^{21} \Rightarrow$

with 2^{42} inp/out pairs can find $k[l_1] \oplus \ldots \oplus k[l_u]$ in time 2^{42}

Roughly speaking: can find 14 key "bits" this way in time 2^{42}

Brute force remaining $56-14=42$ bits in time 2^{42}

Attack time: $\approx 2^{43}$ ($<< 2^{56}$) with 2^{42} random inp/out pairs

Lesson

A tiny bit of linearly leads to a 2^{42} time attack.

$\Rightarrow$ don’t design ciphers yourself !!

(2) Side channel attacks on software AES

Attacker measures the **time** to compute AES128(k,m) for many random blocks m.

- Suppose that the 256-byte S table is not in L1 cache at the start of each invocation

  $\Longrightarrow$ time to encrypt reveals the order in which S entries are accessed

  $\Longrightarrow$ leaks info. that can compromise entire key

Lesson: don’t implement AES yourself !

Mitigation: AES-NI or use vetted software (e.g., BoringSSL)

(3) Quantum attacks

Generic search problem:

Let $f: X \longrightarrow \{0,1\}$ be a function.

Goal: find $x \in X$ s.t. $f(x)=1$.

Classical computer: best generic algorithm time = $O(|X|)$

Quantum computer [Grover '96] : time = $O(|X|^{1/2})$

(requires a long running quantum computation)

Quantum exhaustive search

Given m, c=E(k,m) define

$$f(k) = \begin{cases} 1 & \text{if } E(k,m) = c \\ 0 & \text{otherwise} \end{cases}$$

Grover $\Rightarrow$ quantum computer can find k in time $O(\,|K|^{1/2}\,)$

AES128: quantum key recovery time $\approx 2^{64}$

Adversary has access to a quantum computer $\Longrightarrow$

encrypt data using a cipher with 256-bit keys (AES256)

THE END

Recap

Secure PRF: $F: K \times X \to Y$ and

$\{f(x) = F(k,x) \text{ for } k \leftarrow K\}$ is indist. from random f in Funs[X,Y]

Secure PRP: $E: K \times X \to X$, efficiently invertible, and

$\{\pi(x) = E(k,x) \text{ for } k \leftarrow K\}$ is indist. from random π in Perms[X]

How to use a secure PRF and a secure PRP for encryption?

- One-time key (semantic security): det. CTR-mode
- Many-time key (CPA security):
  nonce-based CBC, nonce-based CTR mode