CS355: Topics in cryptography **Fall 2004**

Assignment #2

Due: Wednesday, Dec. 1, 2004.

Problem 1: (ZK) In class we saw Zero-Knowledge protocols for proving that a number is a quadratic residue modulo N and for proving equality of discrete logarithms. Your goal is to give Zero-Knowledge protocols for the complement languages. Remember to prove soundness, completeness, and zero-knowledge.

a. Give a Zero-Knowledge protocol for the language containing all pairs (N, x) where $x \in \mathbb{Z}_N$ and x is *not* a quadratic residue in $\mathbb{Z}_N$.

b. Let G be a group of prime order q. Give a Zero-Knowledge protocol for the language containing all tuples (g, g^a, h, h^b) where $g, h \in G$ and $a \neq b \bmod q$.

Problem 2: (ZKPK) Let $N = pq$ and let e be a prime, $e \not| \varphi(N)$. Let $v \in \mathbb{Z}_N$ and let $s = v^{1/e} \bmod N$. Consider the following protocol for proving knowledge of s given (N, e, v):

1. Prover picks random $r \in \mathbb{Z}_N$ and sends $t = r^e \in \mathbb{Z}_N$ to verifier.
2. Verifier picks random integer $c \in [1, B]$ and sends c to prover (B is some fixed value).
3. Prover computes $w = s^c \cdot r \in \mathbb{Z}_N$ and sends w to verifier.
4. Verifier accepts only if $w^e = v^c \cdot t$.

a. Prove that the protocol is an honest-verifier ZKPK. Remember to prove completeness, soundness, honest-verifier zero knowledge, and to demonstrate an extractor.

b. Show that there is an efficient malicious prover (who does not know s) that convinces the verifier with probability at least $1/e$.

c. Does it makes sense to chose $B > e$?

Problem 3: (multi-party protocols) For $i = 1, \ldots, n$ suppose that party i has input $a_i \in \mathbb{Z}_p$. Describe an $n - 1$ private protocol for computing $\sum_{i=1}^{n} a_i$. Prove that your protocol is $n - 1$ private (remember to build a simulator for any coalition S of size $|S| < n - 1$).

Problem 4: (two party protocols) Let p be a prime. Suppose user A has an $x \in \mathbb{Z}_p$ and user B has a $y \in \mathbb{Z}_p$. They wish to compute the following function: $f(x, y) = 0$ when $x = y$ and $f(x, y) = 1$ when $x \neq y$, without revealing any other information about x or y. Your goal is to give an efficient solution to this problem in the honest-but-curious settings.

a. Estimate the amount of communication needed for this problem using Yao's garbled circuits method. State your estimate asymptotically as a function of $\log_2 p$. You may assume that we use the Naor-Pinkas OT in (a subgroup) of $\mathbb{Z}_p^*$.

b. Suppose there is a third party who is willing to help. Give an efficient 3-party protocol for computing $f(x, y)$ so that nothing else is revealed to any single party (1-private). Prove 1-privacy by showing a simulator for each party's view of the protocol (the simulator is

given $f(x, y)$ and that party's input).
Hint: Try having the third party pick a random hash function from $\mathcal{H} = \{ax + b \mid a \in \mathbb{Z}_p^*,\ b \in \mathbb{Z}_p\}$.

c. Extra credit: can you suggest 1-private 2-party protocol that is more efficient than Yao's garbled circuit method? Feel free to consult the web.

Problem 5: (Pallier encryption) We discuss the best known additive homomorphic system. Let $N = pq$ and let G be the multiplicative group of integers modulo N^2, i.e. $G = (\mathbb{Z}/N^2\mathbb{Z})^*$. Then G is a group of order $N\varphi(N)$. Let H be the subgroup of G that contains all elements $x \in G$ satisfying $x = 1 \bmod N$. Then H is a subgroup of order N.

a. Show that the discrete log problem in H is easy. That is, show that there is an efficient algorithm that given $g, g^x \in H$ outputs $x \bmod N$.
Hint: note that $g = aN + 1$ for some $a \in \mathbb{Z}$. Use the binomial formula on g^x.

b. Consider the following public-key encryption system:
KeyGen(n): pick two n-bit primes p, q and set $N = pq$. Output the public key N and the private key $d = \varphi(N)$.
Encrypt(N,m): let $g = N + 1 \in G$. For a message $m \in \mathbb{Z}_N$ pick a random $r \in G$ and output the ciphertext $C = g^m r^N \in G$.
Using part (a) show how to decrypt a ciphertext C using the private key $d = \varphi(N)$.
Hint: observe that $C^{\varphi(N)}$ is of order N and therefore in H.

c. Show that given an encryption of x_1 and an encryption of x_2 one can create an encryption of $x_1 + x_2$ sampled from the same the distribution as is produced by the Encrypt algorithm.

d. Let T be the subgroup of G of order $\varphi(N)$, i.e. $T = \{r^N \mid r \in G\}$. Suppose the uniform distribution on T is (t, ϵ) indistinguishable from the uniform distribution on G. Show that the system above is (t, ϵ) semantically secure.