Towards Universal Computation on Ciphertext

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Homomorphic Encryption

Enc. scheme is homomorphic to function f if

- from E[A], E[B], can compute E[f(A,B)]

e.g. f can be $+$, $\times$, $\oplus$, …

Ideally, want f = NAND, or f = $\{+,\times\}$

- Called doubly homomorphic encryption

Can do universal computation on ciphertext!

Why is doubly homomorphic encryption useful?

Gives efficient solutions for many problems.
e.g.

1. 2 party Secure Function Evaluation
   - Alice and Bob have inputs a , b
   - Both want f(a,b) w/o the other learning input
2. Computing on encrypted databases
3. Grid Computing on Sensitive Data

⋮

App: Database Computation

Outsourced server with database containing encrypted data

- User wants to compute function g on encrypted data
  - e.g. data mining, data aggregation

With doubly homomorphic encryption,

- Database encrypted with doubly hom. enc.
- User sends g to server
- Server computes g on encrypted database
- Encrypted result returned to user

App: Distributed Computing on Sensitive Data

Company A has massive amount of data

- Need large computer cluster for computation
  - e.g. DNA analysis, protein folding
- Unwilling to outsource : data leakage

With doubly homomorphic encryption,

- Data encrypted with doubly hom. enc. scheme
- Server sends enc data to cluster computers
- Cluster computes on enc data segments
- Encrypted results returned to server

These applications are pretty cool,

so where can I get a fully homomorphic encryption scheme?

Sorry, it doesn't exist (yet).

- Long standing open problem [RAD78]
- Existing schemes hom. to 1 function
    - E.g. ElGamal ($\times$), Paillier ($+$), GM ($\oplus$)

But some progress …

Main Result

Homomorphic encryption scheme that supports one $\times$ and arbitrary $+$.

- Based on finite bilinear groups with composite order
- Semantic security based on natural decision problem

Enc. Scheme

Keygen(τ):

- G: bilinear group order $n = q_1 q_2$ on ell. curve over F_p.
- Pick rand $g,u \in G$. Set $h = u^{q_2}$.
- PK = (n, G, G_1, e, g, h) SK = q_1

Encrypt(PK, m): $m \in \{1,\ldots,T\}$

- Pick random r from Z_n.
- Output $C = g^m h^r \in G$.

Decrypt(SK, C):

- Let $C^{q_1} = (g^m h^r)^{q_1} = (g^{q_1})^m$; $v = g^{q_1}$
- Output m = Dlog of C^{q_1} base v.

Note: decrypt time is $O(\sqrt{T})$.

Homomorphisms

Given $A = g^a h^r$ and $B = g^b h^s$:

To get encryption of a + b

- pick random $t \in Z_n$
- compute $C = AB \cdot h^t = g^{a+b} h^{r+s+t} \in G$

Bilinear map $e : G \times G \to G_1$.

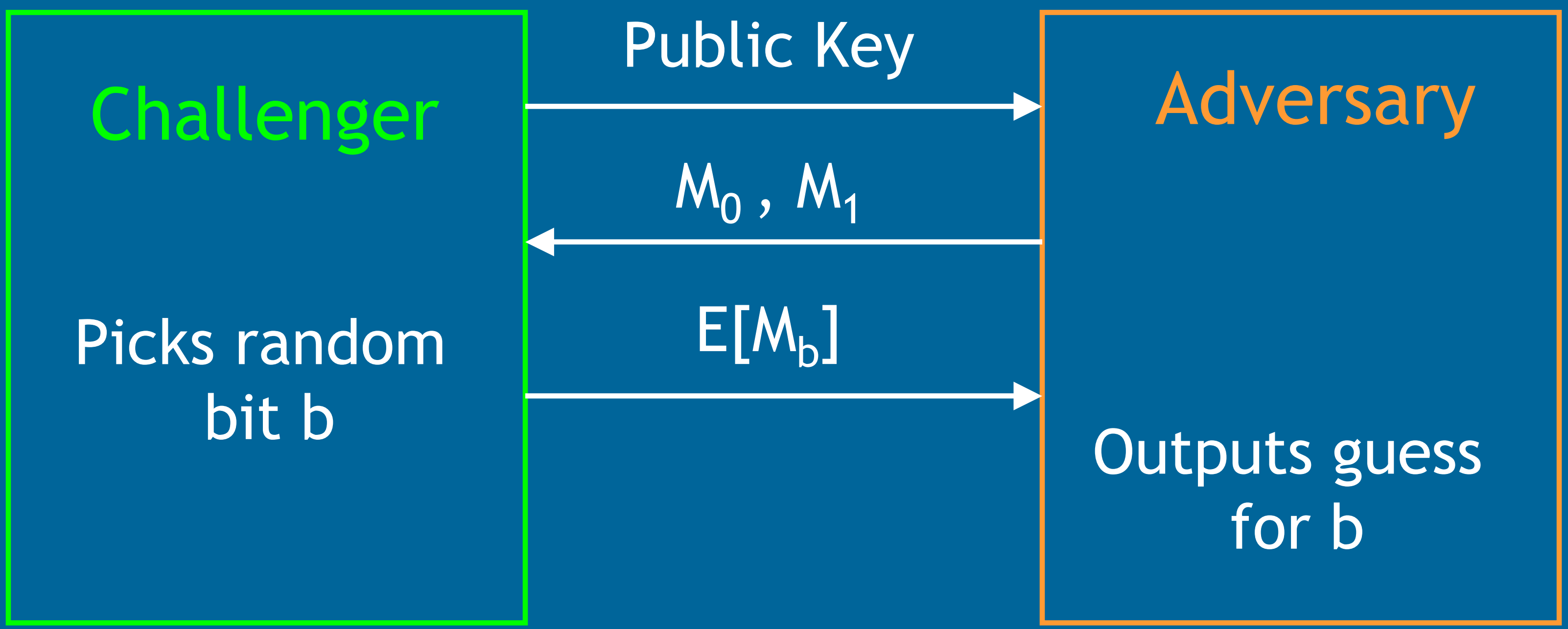
$$e(g^a, g^b) = e(g,g)^{ab} = e(g,g)^{ba} = e(g^b, g^a)$$

To get encryption of a × b

- let $h = g^{\alpha q_2}$, $g_1 = e(g,g)$, $h_1 = e(g,h)$
- pick random $t \in Z_n$
- compute $C = e(A,B) \cdot h_1^{\,t} = g_1^{\,ab} h_1^{\,r'} \in G_1$

Semantic Security

Standard notion of security for enc :



Enc scheme is semantically secure if A guesses b with prob no better than 1/2

- Rules out deterministic encryption

Complexity Assumption

Subgroup assumption:

Gen. rand. bilinear group G of order $n = q_1 q_2$,

then following two distributions indistinguishable:

- x is uniform in G
- x is uniform in q_1—subgroup of G.

Thm: system is semantically secure, unless the subgroup assumption is false.

Applications

what can you do with 1 $\times$ and arbitrary + ?

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1. Evaluate multi-variate polynomials of total degree 2
   - Caveat: result in small set e.g. {0,1}

2. Evaluate 2-DNF formulas $\vee\ (b_{i,1} \wedge b_{i,2})$
   - By arithmetizing 2-DNF formulas to multi-variate poly. with deg 2

1) Evaluating Quadratic Poly.

polynomials of total deg 2

- $x_1 x_2 + x_3 x_4 + \ldots$

- $+, \times$ hom. allow eval. of such poly. on CT
- but to decrypt, result must be in known poly. size interval.
- evaluate dot products

2) 2 Party SFE for 2-DNF

Bob

$A = (a_1,\ldots,a_n) \in \{0,1\}^n$

Alice

$\phi(x_1,\ldots,x_n) = \vee^k_{i=1}(y_{i,1} \wedge y_{i,2})$ s.t. $y_{i,*} \in \{x_1, \neg x_1, \ldots, x_n, \neg x_n\}$.

Get Arithmetization Φ:

- replace $\vee$ by $+$, $\wedge$ by $\times$, $\neg x_i$ by $(1- x_i)$.
- Φ is poly. with total deg 2!

2-DNF Protocol (Semi-Honest)

Bob	Alice
$A = (a_1,\ldots,a_n)$	$\phi(x_1,\ldots,x_n) = \vee^k_{i=1}(y_{i,1} \wedge y_{i,2})$
	Φ = arith. of ϕ

Invoke Keygen(τ)

Encrypt A

PK, $E[a_1],\ldots,E[a_n]$ →

← $E[r \cdot \Phi(A)]$

Eval. $E[r \cdot \Phi(A)]$ for random r

If decrypt = 0, emit 0. Else, 1.

Bob's Security: Alice cannot distinguish bet. Bob's possible inputs — from semantic security of E.

Alice's Security: Bob only knows if A satisfies $\phi()$ — by design, Bob output distrib. depends only on this.

Concrete applications

1. Improve basic step in Kushilevitz-Ostrovsky PIR protocol from $\sqrt{n}$ to $\sqrt[3]{n}$

2. Gadget: "check" if CT contains 1 of 2 values.
   - Most voter efficient E-voting scheme
   - Universally verifiable computation

PIR/SPIR

Bob: wants D(R,S)

Set assignment A:

$x_R = y_S = 1$,

$x_i = x_j = 0$

for $i \neq R$, $j \neq S$

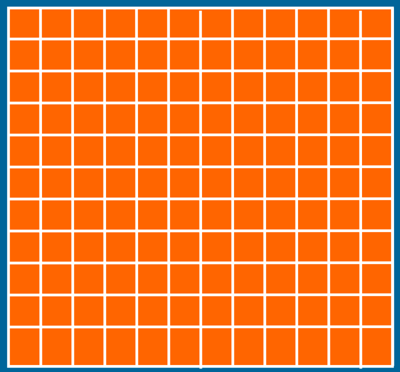
Do 2-DNF SFE

with A and ϕ

Get $\phi(A) = D(R,S)$

Database D

$\sqrt{n}$ $|D| = n$



$\sqrt{n}$

D uses 2-DNF

$\phi(x_1,\ldots,x_{\sqrt{n}}, y_1,\ldots,y_{\sqrt{n}})$

$= \vee_{D(i,j)=1} (x_i \wedge y_j)$

Comm. Complexity = $O(\tau \cdot \sqrt{n})$ [$O(\tau \cdot \sqrt[3]{n})$ balanced]

Alternative scheme — each db entry $O(\log n)$ bits

Conclusions

Adding even limited additional homomorphism has many uses.

Open Problems:

- Extend encryption scheme to
  1. efficiently handle arbitrary messages
  2. arbitrary # of multiplications
- Find n-linear maps
  - allow eval. of polynomials with total deg n

Questions?