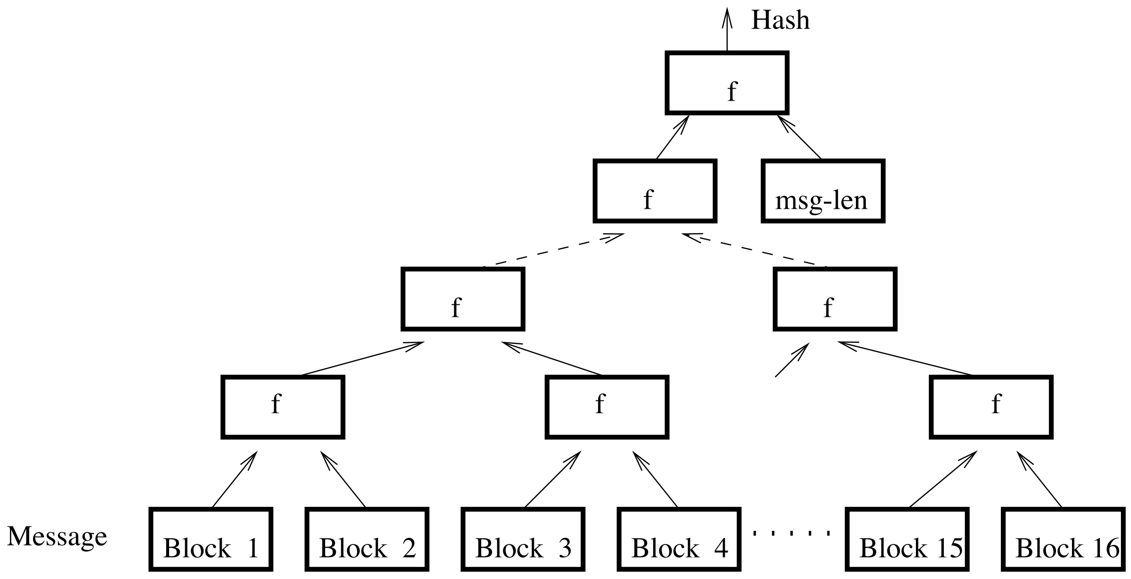
**CS255: Cryptography and Computer Security** **Winter 2013**

# Assignment #2

Due: 5pm Friday, Feb. 22, 2013.

**Problem 1.** Merkle hash trees.

Merkle suggested a parallelizable method for constructing hash functions out of compression functions. Let $f$ be a compression function that takes two 512 bit blocks and outputs one 512 bit block. To hash a message $m$ one uses the following tree construction:



For similicity, let's assume that the number of blocks in $m$ is always a power of 2.

**a.** Prove that if one can find a collision for the resulting hash function then one can find collisions for the compression function.

**b.** Show that if the msg-len block is eliminated (e.g. the contents of that block is always set to 0) then the construction is not collision resistant.

**Problem 2.** In the lecture we saw that Davies-Meyer is often used to convert an ideal block cipher into a collision resistant compression function. Let $E(k, m)$ be a block cipher where the message space is the same as the key space (e.g. 128-bit AES). Show that the following methods do not work:

$$f_1(x, y) = E(y, x) \oplus y \quad \text{and} \quad f_2(x, y) = E(x, x) \oplus y \oplus x$$

That is, show an efficient algorithm for constructing collisions for $f_1$ and $f_2$. Recall that the block cipher $E$ and the corresponding decryption algorithm $D$ are both known to you.

**Problem 3.** Suppose we implement the CBC-MAC using a PRP $(E, D)$. Show that for any key $k$ the function $H(m) := \text{rawCBC}_E(k, m)$ is not collision resistant. That is, for an arbitrary key $k$ show how to construct distinct $m$ and $m'$ such that $\text{rawCBC}_E(k, m) = \text{rawCBC}_E(k, m')$. Note that here $k$ is public.

**Problem 4.** Suppose user $A$ is broadcasting packets to $n$ recipients $B_1, \dots, B_n$. Privacy is not important but integrity is. In other words, each of $B_1, \dots, B_n$ should be assured that the packets he is receiving were sent by $A$. User $A$ decides to use a MAC.

**a.** Suppose user $A$ and $B_1, \dots, B_n$ all share a secret key $k$. User $A$ MACs every packet she sends using $k$. Each user $B_i$ can then verify the MAC. Using at most two sentences explain why this scheme is insecure, namely, show that user $B_1$ is not assured that packets he is receiving are from $A$.

**b.** Suppose user $A$ has a set $S = \{k_1, \dots, k_m\}$ of $m$ secret keys. Each user $B_i$ has some subset $S_i \subseteq S$ of the keys. When $A$ transmits a packet she appends $m$ MACs to it by MACing the packet with each of her $m$ keys. When user $B_i$ receives a packet he accepts it as valid only if all MAC's corresponding to keys in $S_i$ are valid. What property should the sets $S_1, \dots, S_n$ satisfy so that the attack from part (a) does not apply? We are assuming all users $B_1, \dots, B_n$ are sufficiently far apart so that they cannot collude.

**c.** Show that when $n = 10$ (i.e. ten recipients) the broadcaster $A$ need only append 5 MAC's to every packet to satisfy the condition of part (b). Describe the sets $S_1, \dots, S_{10} \subseteq \{k_1, \dots, k_5\}$ you would use.

**Problem 5.** CBC padding attack. Recall that when using CBC mode, TLS pads messages to a multiple of the block length by appending a $t$ byte pad for a suitable value of $t$ and all bytes of the pad are set to $t - 1$. For example, if a 2 byte pad is needed, TLS appends $(1, 1)$ to the plaintext prior to CBC encryption. The recipient, after decrypting the CBC chain, checks that the pad has the correct format and if not rejects the ciphertext. A bug in older versions of OpenSSL lets the attacker learn if ciphertext rejection happened due to a bad pad.

Now, suppose an attacker intercepts a target ciphertext $c_{full}$. The attacker deletes the last block of $c_{full}$ thereby deleting any padding blocks. Let $c$ be the resulting truncated ciphertext and let $m$ be the result of decrypting this $c$ using CBC decryption. Your goal is to show that this OpenSSL bug can let the attacker test if the last of byte of $m$ is equal to some byte $g$ of the attacker's choosing. Using $c$, construct a ciphertext $c'$ that has the following property: when $c'$ is sent to the server, the decryption of $c'$ will end with a valid pad if the last byte of $m$ is equal to $g$ and will end with an invalid pad (with high probability) otherwise. By sending $c'$ to the server, the attacker can therefore learn if $m$ ends with $g$.

note: In principle, the attacker can repeat this experiment for all 256 values of $g$ until a match is found. He then learns the last byte of $m$. However, TLS tears down the connection and renegotiates a new key when a pad error occurs and therefore this typically cannot be applied to TLS. Nevertheless, by injecting Javascript into an insecure connection the attacker can cause the message $m$ to be sent over and over on new TLS connections. Each such transmission gives the attacker an opportunity to test one value of $g$. This clever attack lets an attacker learn the value of a user's secret session cookie one byte at a time even if the cookie is only transmitted over HTTPS.

**Problem 6.** Conference key setup.

Parties $A_1, \dots, A_n$ and $B$ wish to generate a secret conference key. All parties should know the conference key, but an eavesdropper should not be able to obtain any information about the key. They decide to use the following variant of Diffie-Hellman: there is a public prime $p$ and a public element $g \in \mathbb{Z}_p^*$ of order $q$ for some large prime $q$ dividing $p-1$. User $B$ picks a secret random $b \in [1, q-1]$ and computes $y = g^b \in \mathbb{Z}_p^*$. Each party $A_i$ picks a secret random $a_i \in [1, q-1]$ and computes $x_i = g^{a_i} \in \mathbb{Z}_p^*$. User $A_i$ sends $x_i$ to $B$. User $B$ responds to party $i$ by sending $z_i = x_i^b \in \mathbb{Z}_p^*$.

**a.** Show that party $i$ given $z_i$ (and $a_i$) can determine $y$.

**b.** Explain why (a hash of) $y$ can be securely used as the conference key. Namely, explain why at the end of the protocol all parties $A_1, \dots, A_n$ and $B$ know $y$ and give a brief informal explanation why an eavesdropper cannot determine $y$.

**c.** Prove part (b). Namely, show that if there exists an efficient algorithm $\mathcal{A}$ that given the public values in the above protocol, outputs $y$, then there also exists an efficient algorithm $\mathcal{B}$ that breaks the Computational Diffie-Hellman assumption in the subgroup of $\mathbb{Z}_p^*$ generated by $g$. Use algorithm $\mathcal{A}$ as a subroutine in your algorithm $\mathcal{B}$. Note that algorithm $\mathcal{A}$ takes as input a triple $(g, g^x, g^y)$ and outputs $g^{x/y}$ while algorithm $\mathcal{B}$ takes as input a triple $(g, g^x, g^y)$ and outputs $g^{xy}$

**Problem 7.** Private equality test. Suppose Alice has a secret number $a \in \mathbb{Z}_q$ and Bob has a secret number $b \in \mathbb{Z}_q$, for some prime $q$. They wish to design a private equality protocol, namely a protocol such that at the end, if $a = b$ Alice learns that fact, but if $a \neq b$ then Alice learns nothing else about $b$. Either way Bob learn nothing about $a$. Think of $a$ and $b$ as hashes of a file. The two want to test if they have the same file without leaking any other information about the contents of their files.

Let $\mathcal{E}$ be a public key encryption system with message space $\mathbb{Z}_q$. $\mathcal{E}$ satisfies the following property: given pk and $c_1 = E(\text{pk}, x)$ and $c_2 = E(\text{pk}, y)$ for $x, y$ in $\mathbb{Z}_q$ it is possible to create a new ciphertext $c$ that is a fresh random encryption of $x + y$, namely $c = E(\text{pk}, x+y)$ and $c$ is independent of $c_1$ and $c_2$. An encryption scheme with this property is said to be *additively homomorphic.* Now, Alice proposes the following protocol:

- Alice generates a pk/sk pair for $\mathcal{E}$ and sends pk and $c_1 := E(\text{pk}, a)$ to Bob.
- Bob chooses random $s \xleftarrow{R} \mathbb{Z}_q^*$ and computes the encryption of $E(\text{pk},\ s(a - b))$. Call the resulting ciphertext $c_2$. Bob sends $c_2$ back to Alice.

**a.** Explain how Bob computes $c_2$ given pk, $c_1$ and $s$.
**Hint:** you will need to use a variant of the repeated squaring algorithm.

**b.** Explain how Alice uses $c_2$ and her secret key to test if $a = b$. If $a \neq b$, explain why Alice learns nothing about $b$ other than the fact that $a \neq b$.

**c.** Let $\mathbb{G}$ be a group of order $q$ where the Decision Diffie Hellman problem is hard. Let $g$ be a generator of $\mathbb{G}$ and let $\mathcal{E}$ be the following variant of ElGamal encryption: the public key is $\text{pk} = (g, h = g^d)$ and the secret key is $d$. The "encryption" of a message $a \in \mathbb{Z}_q$ is defined as:

$$E(\text{pk}, a) = (g^r, h^{r+a}) \quad \text{where } r \xleftarrow{R} \mathbb{Z}_q$$

Show that given $E(\text{pk}, a)$ Alice can use her secret key $d$ to compute $h^a$. Note that this system is not really an encryption system since Alice cannot fully recover the plaintext $a$ from a given ciphertext.

**d.** Explain how to instantiate the protocol above using the system from part (c). Describe exactly what message Alice would send to Bob, how Bob would respond, and how Alice would learn the comparison result.