**CS154: Introduction to Automata and Complexity Theory** **Fall 1997**

# Final Exam

**Instructions:** Answer five of the following six questions. Each question is worth two points.

**Problem 1** Prove that the following language is not R.E.

$$L = \left\{ < M_1 >< M_2 > \;\middle|\; |L(M_1)| > |L(M_2)| \right\}$$

**Problem 2**

1. Prove that if $P = NP$ then *every* language $L \in NP$ is $NP-$complete, except $L = \emptyset$ and $L = \Sigma^*$.
2. Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ is called a $K$-Moon-Clique if $|S| = K$, $S$ contains a $k-1$ clique and another vertex $v$ which is not connected to any of the vertices of the $k-1$ clique. (the $k-1$ clique is the sun and $v$ is the moon).

   For example the graph $u - v\ w$ is a 3-SW clique: $\{u, v\}$ is the sun and $w$ is the moon.

   Prove that if $P \neq NP$ then the language $L_{SW} = \{(G, K) \mid G$ a graph with a $K-$Moon-Clique $\}$ is not in $P$.

**Problem 3** Answer the following True/False questions. Justify your answer by giving counterexamples or proof sketches where necessary.

1. All $NP-$complete languages are recursive.
2. Suppose $P = NP$. Then any R.E. language is also in $P$.
3. If $L \in P$ then $L \in NTIME(n^k)$ for some $k$.
4. If $f(n) = O(n)$ and $f(n) = \Omega(n)$, then $f(n) = \theta(n)$.
5. If $L_1, L_2 \in NP$ then so is $L_1 \cap L_2$.
6. If a language $L_1$ is mapping reducible to $L_2$, i.e. $L_1 \leq_m L_2$, and $L_2$ is undecidable then so is $L_1$.
7. If a language $L_1$ is polynomial time reducible to $L_2$, i.e. $L_1 \leq_P L_2$, and $L_1$ is undecidable then so is $L_2$.
8. If a language $L_1$ is mapping reducible to $L_2 \in NP$, i.e. $L_1 \leq_m L_2$, and $L_1$ is $NP-$complete then so is $L_2$. For your proof or counterexample you may assume $P \neq NP$.

**Problem 4** Consider the following language:

$$L_{even} = \left\{ w \;\middle|\; \begin{array}{l} w \text{ is the binary representation} \\ \text{of an even positive integer} \end{array} \right\}$$

1. Prove that $L_{even}$ is decidable by *precisely* describing a TM for deciding it. A figure is sufficient.
2. Prove that the language $L = \{< M > \mid L(M) = L_{even}\}$ is undecidable.

**Problem 5** Define the language

$$L_{composites} = \{< N > \mid N \text{ is a composite integer}\}$$

where $< N >$ is the binary representation of $N$.

1. Prove that $L_{composites} \in NP$.
2. Consider the following algorithm for $L_{composites}$: given $N$ the algorithm divides $N$ by all integers $k = 2, \ldots, N-1$. If $N$ is divisible by some $k$ the algorithm accepts. Otherwise the algorithm rejects. This algorithm runs in polynomial time in $N$. Does it prove that $L_{composites}$ is in $P$?

**Problem 6**

Hitting Set: Let $S = \{1, \ldots, n\}$ be a finite set. Let $C_1, \ldots, C_m \subseteq S$ be subsets of $S$ and $K \leq n$ an integer. A subset $S' \subseteq S$ is a $K$-Hitting Set if $|S'| \leq K$ and $S'$ contains at least one element from each subset $C_i$.

For example, if $S = \{1, 2, 3, 4, 5\}$ and $C_1 = \{1, 2\}$; $C_2 = \{2, 4, 5\}$; $C_3 = \{3, 4\}$ then $S' = \{2, 3\}$ is a 2-Hitting Set since it hits all three subsets.

Prove that the language

$$L = \{< S, C_1, \ldots, C_m, K > \mid S \text{ contains a } K\text{-Hitting Set}\}$$

is $NP-$Complete.

Hint: Use Vertex Cover.