Quicksort

- Quicksort(A, p, r)
  while p < r
    q = partition(A, p, r)
    quicksort(A, p, q - 1)
    quicksort(A, q + 1, r)
  end

- To simplify, assume distinct elements.
  » Lucky - always an even split: \( T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = \Theta(n \log n) \)
  » Unlucky:
    \( T(n) = T(0) + T(n-1) + \Theta(n) \Rightarrow T(n) = \Theta(n^2) \)

- How to avoid bad case?
  » Partitioning around middle element does not work!
  » Idea: partition around a random element.

Randomized Algorithms

- Algorithm can “toss coins”.
- No specific input leads to worst-case behavior.
- Distinction between randomized algorithms and random data!
Quick review of probability

- Sample space $S$ of “elementary events”.
  - Example: 36 ways of how 2 dice can fall.
- Event $A \subseteq S$, Eg. “roll 3 with 2 dice”.
- Probability distribution: $P : A \rightarrow [0,1]$, \(2^{10}\) values
- Properties:
  \[
P(A) \geq 0, P(S) = 1
  \]
  \[
P(A \cup B) = P(A) + P(B) \text{ if } A \cap B = \emptyset
  \]

Example

- 2 dice example:
  \[
  S = \{(1,1),(1,2),(2,1),\ldots,(6,6)\}, |S| = 36
  \]
  \[
  (5,6) \neq (6,5)!!
  \]
  \[
  \text{Event roll 4: } \{1,3),(2,2),(3,1)\}
  \]
  \[
  \Pr[A] = \frac{|A|}{36} = \frac{3}{36}
  \]
- Simple case of “inclusion/exclusion”:
  \[
  \Pr[A \cup B] = P(A) + P(B) - P(A \cap B)
  \leq P(A) + P(B)
  \]
### Discrete Random Variable

- **Definition:**
  \[ X: S \rightarrow R \]
  - \( X = i \) \( \iff \) \( \{ s \in S | X(s) = i \} \)
  - Example: Uniform distr, 2 dice: \( \Pr[X = 5] = 4/36 \)

- **Expected value:**
  \[
  E[X] = \sum i \Pr[X = i]
  \]

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\[
E[X] = \frac{252}{36} = 7
\]

### Linearity of expectation

- **E[aX+bY]=aE[X]+bE[Y]**

- **Example:**
  - \( X \) - outcome of first, \( Y \) - outcome of second.
  - \( E[X] = E[Y] = \frac{1+2+3+\ldots+6}{6} = 3.5 \)
  - \( E[X+Y] = 7 \), as before!

- **Independence:**
  - \( X \) & \( Y \) independent \( \iff \) \( \forall x, y: \Pr[X = i, Y = j] = \Pr[X = i] \Pr[Y = j] \)
  - \( E[X \cdot Y] = E[X]E[Y] \)
Conditional Probability

- **Definition**: \( \Pr[X = i | Y = j] = \frac{\Pr[X = i, Y = j]}{\Pr[Y = j]} \)

- **Conditional expectation**:
  
  \[
  E_i[E[X | Y]] = \sum_i \Pr[Y = i] E_i[X | Y = i] = \sum_i \Pr[Y = i] \sum_j \Pr[X = j | Y = i] \times E[X | Y = i]
  \]

  \[
  = \sum_i \Pr[X = j] \sum_j \Pr[X = j | Y = i] \times E[X | Y = i]
  \]

  \[
  = \sum_i \Pr[X = j] \sum_j \Pr[X = j | Y = i]
  \]

  \[
  = \sum_i \Pr[X = j | Y = i] \sum_j \Pr[X = j]
  \]

  \[
  = E[X]
  \]

One of the most useful properties
Conditional expectation example

- Consider 1-dice toss.

- Let X be result of the toss, and Y be the event that the result is above 2. (Y=1 if above 2, Y=0 otherwise.)

- Condition on Y.
  Note that Pr[Y=0]=2/6, Pr[Y=1]=4/6.

\[
E[X|Y=0] = \sum Pr[X=x|Y=0] = \frac{1}{6} \cdot \frac{2}{6} + \frac{1}{6} \cdot \frac{3}{2} = \frac{1+\frac{3}{2}}{2} = \frac{5}{4}
\]

\[
E[X|Y=1] = \sum Pr[X=x|Y=1] = \frac{1}{6} \cdot \frac{3}{6} + \frac{1}{6} \cdot \frac{4}{2} = \frac{3+4}{6} = \frac{7}{2}
\]

\[
E[X] = E[X|Y=0]Pr[Y=0] + E[X|Y=1]Pr[Y=1] = \frac{5}{4} \cdot \frac{2}{6} + \frac{7}{2} \cdot \frac{4}{6} = 3.5
\]

Back to Quicksort

- Partition around a randomly chosen element and let T(n) be the expected time to sort.

- Consider the case where the partition is (k, n-k-1).
  In this case, the expected time to terminate is:

\[
T(k) + T(n-1-k) + \Theta(n)
\]

- Condition on k being a specific value.
  Note that any value of k, from 0 to n-1 is equally likely.

\[
T(n) = \sum Pr((i, n-i-1) \text{ split}) T(n| (i, n-i-1) \text{ split})
- \frac{1}{2} \sum [T(k) + T(n-1-k) + \Theta(n)]
- \frac{1}{2} \sum [T(k) + \Theta(n)]
\]
Solving the recurrence

We will try to prove that \( T(n) \leq an \lg n + b \)

First, choose \( b \) large enough to satisfy: \( T(1) \leq a \lg 1 + b = b \)

Inductive step:

\[
T(n) = 2 \sum_{k=1}^{\lfloor n/2 \rfloor} T(k) + \Theta(n) \leq 2 \sum_{k=1}^{\lfloor n/2 \rfloor} (ak \lg k + b) + \Theta(n)
\]

\[
= \frac{2}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} k \lg k + \frac{2}{n} nb + \Theta(n)
\]

Need to prove that this is \( \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \)

Note that using \( \sum_{k=1}^{n} k \lg k \leq n^2 \lg n \) is not enough !

Technical lemma

\( n^2 \lg n \) bound is trivial. Need a stronger bound

\[
\sum_{k=1}^{n} k \lg k = \sum_{k=1}^{\lfloor n/2 \rfloor} k \lg k + \sum_{k=\lfloor n/2 \rfloor}^{n} k \lg k
\]

\[
\leq \lg n \sum_{k=1}^{\lfloor n/2 \rfloor} k - \sum_{k=\lfloor n/2 \rfloor}^{n} k
\]

\[
\leq \lg n \sum_{k=1}^{\lfloor n/2 \rfloor} k - \sum_{k=1}^{n} k
\]

\[
\leq \lg n \frac{n(n-1)}{2} - \frac{n(n+1)}{2}
\]

\[
\leq \frac{1}{2} n^2 \lg n - \frac{n^2}{8}
\]

HW: We proved \( O \), now prove \( \Omega \).