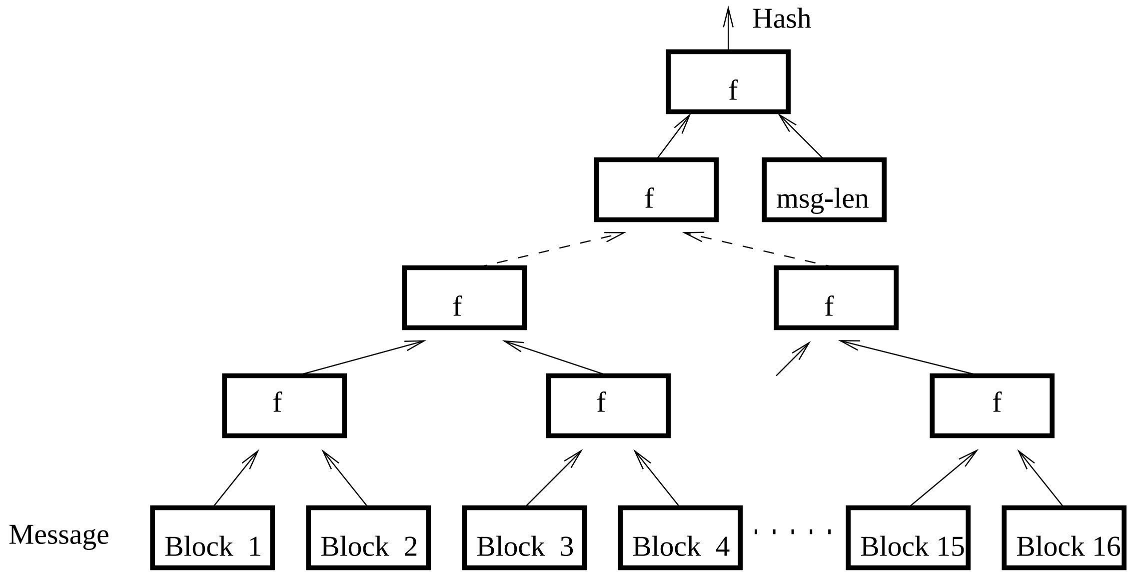
CS255: Cryptography and Computer Security — Winter 2000

# Assignment #3

Due: Monday, February 28th, 2000.

**Problem 1** Merkle hash trees.
Merkle suggested a parallelizable method for constructing hash functions out of compression functions. Let $f$ be a compression function that takes two 512 bit blocks and outputs one 512 bit block. To hash a message $M$ one uses the following tree construction:




Prove that if one can find a collision for the resulting hash function then one can find collisions for the compression function.

**Problem 2** In this problem we explore the different ways of constructing a MAC out of a non-keyed hash function. Let $h : \{0,1\}^* \to \{0,1\}^b$ be a hash function constructed by iterating a collision resistant compression function using the Merkle-Damgård construction.

1. Show that defining $MAC_k(M) = h(k \parallel M)$ results in an insecure MAC. That is, show that given a valid text/MAC pair $(M, H)$ one can efficiently construct another valid text/MAC pair $(M', H')$ without knowing the key $k$.
2. Recall that in the Merkle-Damgård iterated construction one uses a fixed Initial Value IV as the initial chaining variable. Show that setting the $IV$ to be the secret key $k$ results in an insecure MAC.

3. Consider the MAC defined by $MAC_k(M) = h(M \parallel k)$. Show that in expected time $O(2^{b/2})$ it is possible to construct two messages $M$ and $M'$ such that given $MAC_k(M)$ it is possible to construct $MAC_k(M')$ without knowing the key $k$.

4. Give a short high level argument to show why the envelope method for constructing a MAC out of a hash function produces a secure MAC.

**Problem 3** Rabin suggested a signature scheme very similar to RSA signatures. In its simplest form, the public key is a product of two large primes $N = pq$ and the private key is $p$ and $q$. The signature $S$ of a message $M \in \mathbb{Z}_N$ is the square root of $M$ modulo $N$. For simplicity, assume that the messages $M$ being signed are always quadratic residues modulo $N$. To verify the signature, simply check that $S^2 = M \bmod N$. Note that we did not include any hashing of $M$ prior to signing. Show that a chosen message attack on the scheme can result in a total break. More precisely, if an attacker can get Alice to sign messages chosen by the attacker then the attacker can factor $N$.
**Hint:** recall that a quadratic residue modulo $N = pq$ has four square roots in $\mathbb{Z}_N$. First show that there are two square roots of $M$ that enable the attacker to factor $N$ (use the fact that gcd's are easy to compute). Then show how using a chosen message attack the attacker can get a hold of such a pair of square roots. Note that proper hashing prior to signing prevents this attacks.

**Problem 4** Suppose Alice and Bob share a secret key $k$. A simple proposal for a MAC algorithm on fixed length messages is as follows: given a message $M$ do: (1) compute 128 different parity bits of $M$ (i.e. compute the parity of 128 different subsets of the bits of $M$), and (2) DES encrypt the resulting 128-bit checksum using $k$. Naively, one could argue that without knowing $k$ an attacker cannot compute the MAC of a message $M$. Show that this proposal is flawed. Note that the algorithm for computing the 128-bit checksum is public.
Hint: show that an attacker can carry out an existential forgery given one valid message/MAC pair. Use linear algebra modulo 2.

**Extra Credit** Recall that in the ElGamal signature scheme, a signature is of the form $(a, b)$ where $b \in Z_q$ and $a$ is an integer. In lecture, we glossed over the fact that $a$ is required to be less than $p$. Show that without this restriction, one can forge signatures for any message.
**Hint:** ElGamal says to find $(a, b)$ such that $y^a a^b = g^M$. First show how to find $(b, c, d)$ such that $y^d c^b = g^M$.]