<u>CS255</u>: Winter 2025

CPA Security:
How to use a key multiple times

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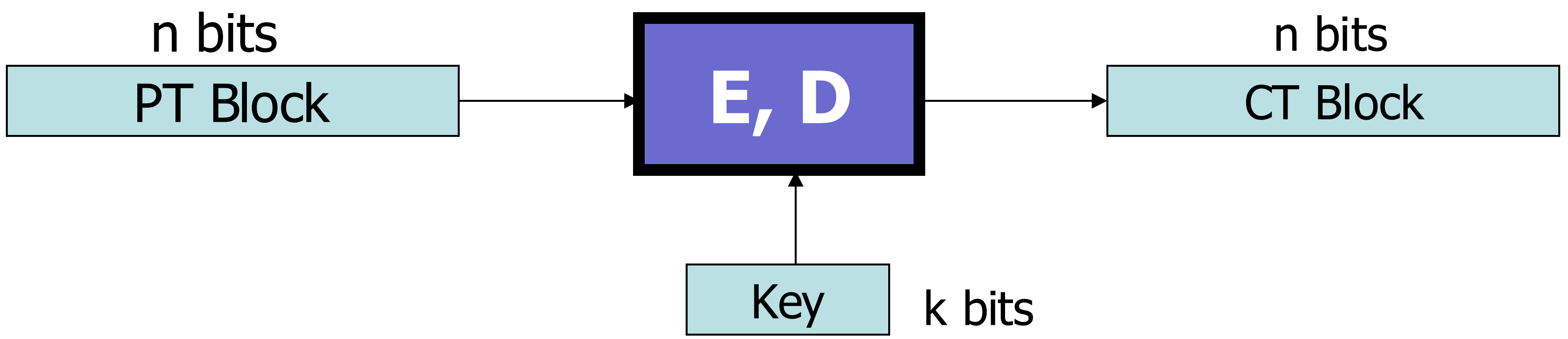
Quick Recap

A **block cipher** is a pair of efficient algs. (E, D):



Canonical examples:

- **AES**: n=128 bits, k = 128, 192, 256 bits
  (hardware support for many blocks in parallel)

- **3DES**: n= 64 bits, k = 168 bits (historical)

Abstract block ciphers: PRFs and PRPs

PRF: an efficiently computable $F: K \times X \to Y$

PRP: (a.k.a a block cipher) $E: K \times X \to X$

is a PRF, such that

- for all $k \in K$: the function $E(k, \cdot)$ is one-to-one,
- there is an "efficient" inversion algorithm $D(k, x)$.

Secure PRF (resp. PRP):

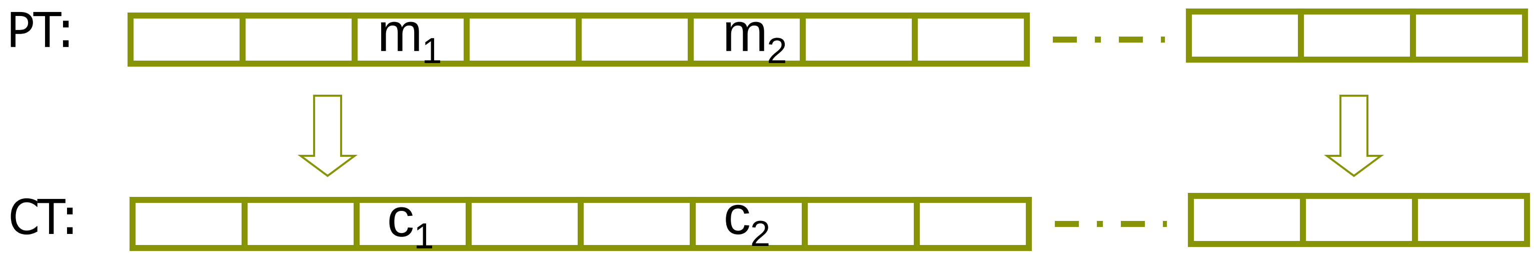
the uniform distribution on $S_F \coloneqq \{ F(k, \cdot) : k \in K \}$

is **indistinguishable by queries** from

the uniform distribution on $\mathrm{Funs}[X, Y]$ $(\text{resp. } \mathrm{Perms}[X])$.

ECB: Incorrect use of a PRP

Electronic Code Book (ECB):



Problem:

– if $m_1 = m_2$ then $c_1 = c_2$

How to use a block cipher?

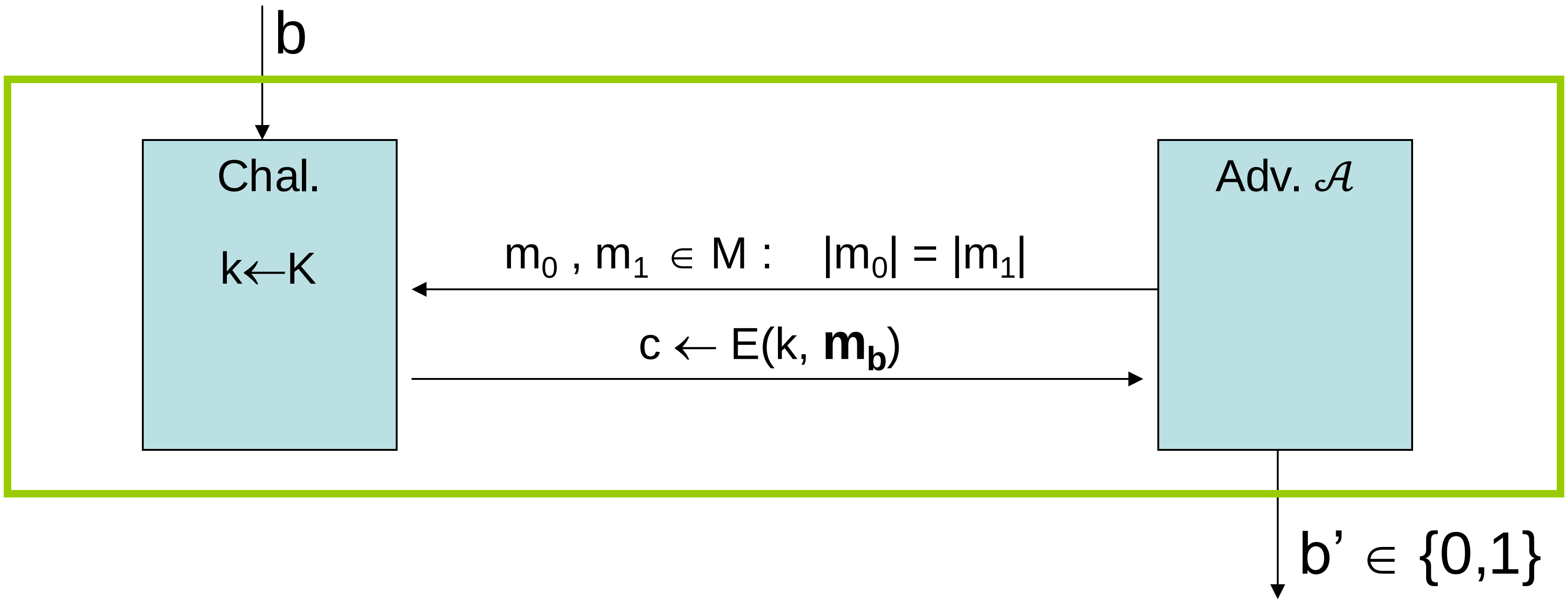
Modes of Operation for One-time Use Key

Example application:

Encrypted email. New key for every message.

Semantic Security for a one-time key

- $\mathbb{E} = (E,D)$ a cipher defined over (K,M,C)
- For $b=0,1$ define $EXP(b)$ as:



- Def: $\mathbb{E}$ is sem. sec. for one-time key if for all "efficient" $\mathcal{A}$:

$$\mathrm{Adv}_{SS}[\mathcal{A},\mathbb{E}] = \Big|\Pr[EXP(0)=1] - \Pr[EXP(1)=1]\Big|$$

is "negligible."

A Semantically Secure Scheme

Deterministic counter mode from a PRF

$$F: \ K \times \{0,1,\ldots,L\} \rightarrow \{0,1\}^n$$

$E_{DETCTR}(k,m) =$

m[0]	m[1]	…	m[L]

$\oplus$

F(k,0)	**F(k,1)**	…	**F(k,L)**

indist. from a OTP

c[0]	c[1]	…	c[L]

⇒ Stream cipher built from PRF (e.g. AES)

How to use a block cipher?

Modes of Operation for Many-time Key

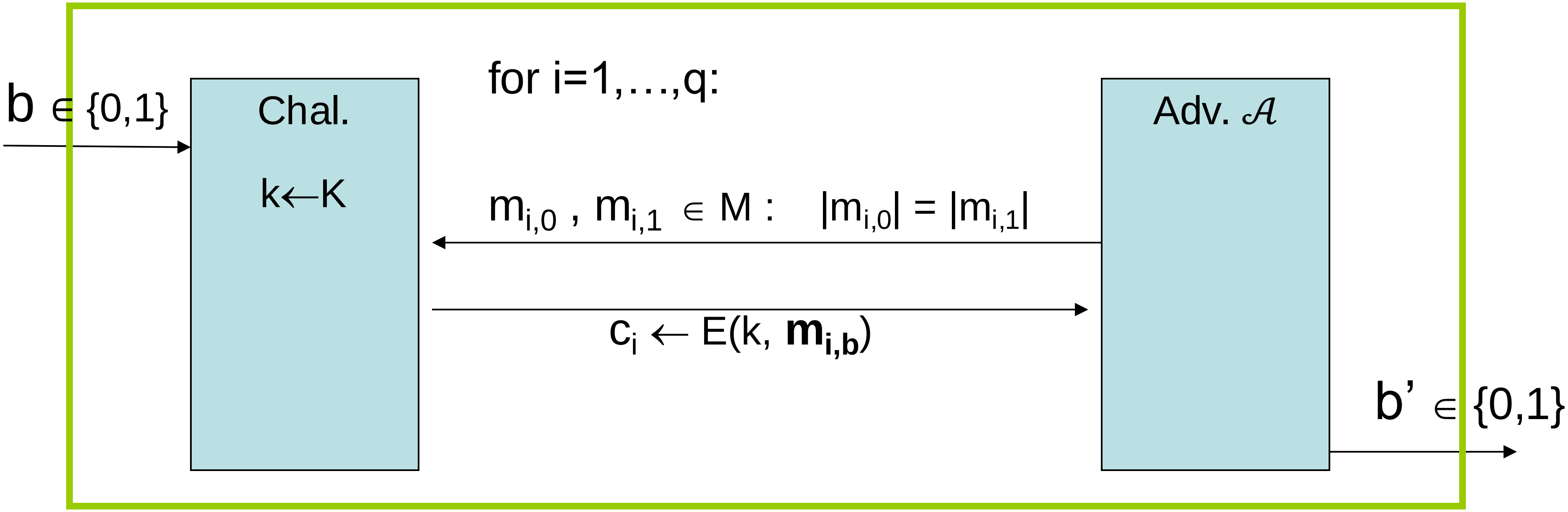
<u>Example applications</u>:

1. File systems: Same AES key used to encrypt many files.
2. IPsec: Same AES key used to encrypt many packets.

Semantic Security for many-time key (CPA security)

Cipher $\mathbb{E} = (E,D)$ defined over (K,M,C).

For $b=0,1$ define $EXP(b)$ as:



if adv. wants $c = E(k, m)$ it queries with $m_{j,0} = m_{j,1} = m$

Def: $\mathbb{E}$ is sem. sec. under CPA if for all "efficient" $\mathcal{A}$:

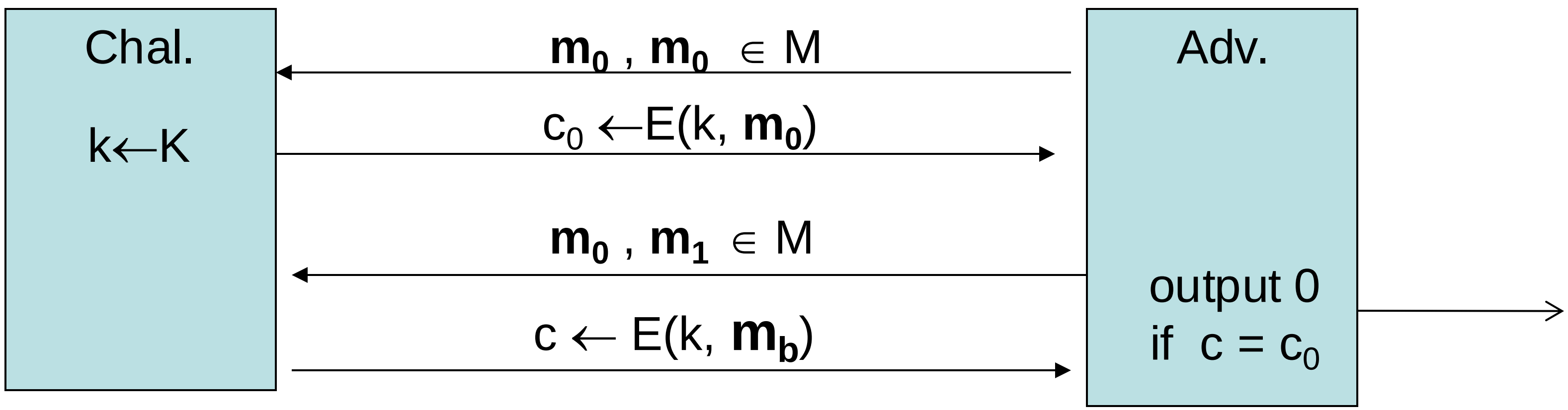
$$\mathrm{Adv}_{CPA}[\mathcal{A},\mathbb{E}] = \big|\Pr[EXP(0)=1] - \Pr[EXP(1)=1]\big|$$

is "negligible."

Security for many-time key

<u>Fact:</u> stream ciphers are insecure under CPA.

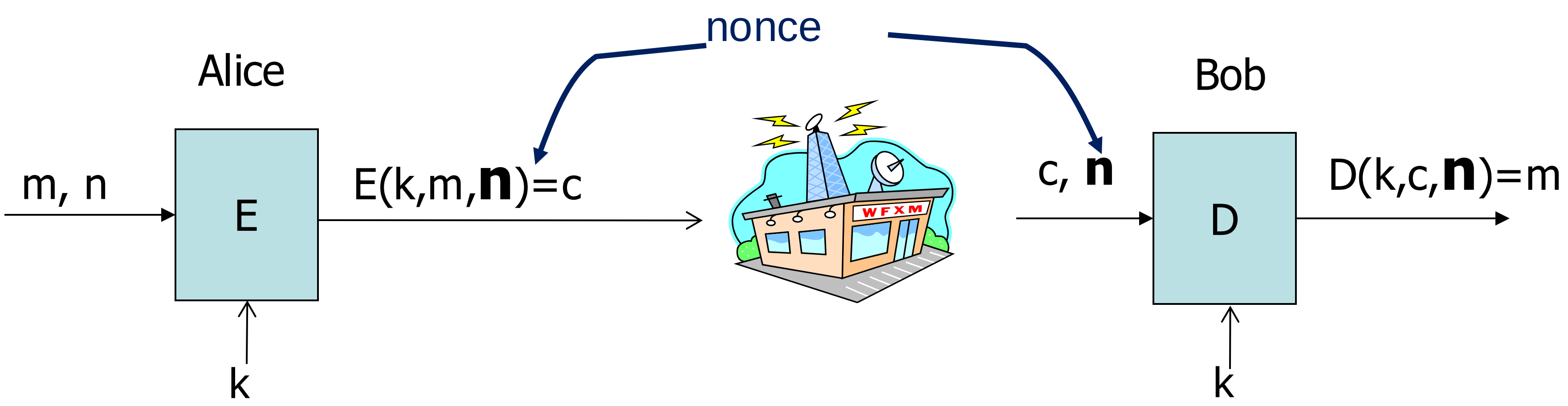
- More generally: if E(k,m) always produces same ciphertext, then cipher is insecure under CPA.



If secret key is to be used multiple times $\Rightarrow$

given the same plaintext message twice,
the encryption alg. must produce different outputs.

Nonce-based Encryption



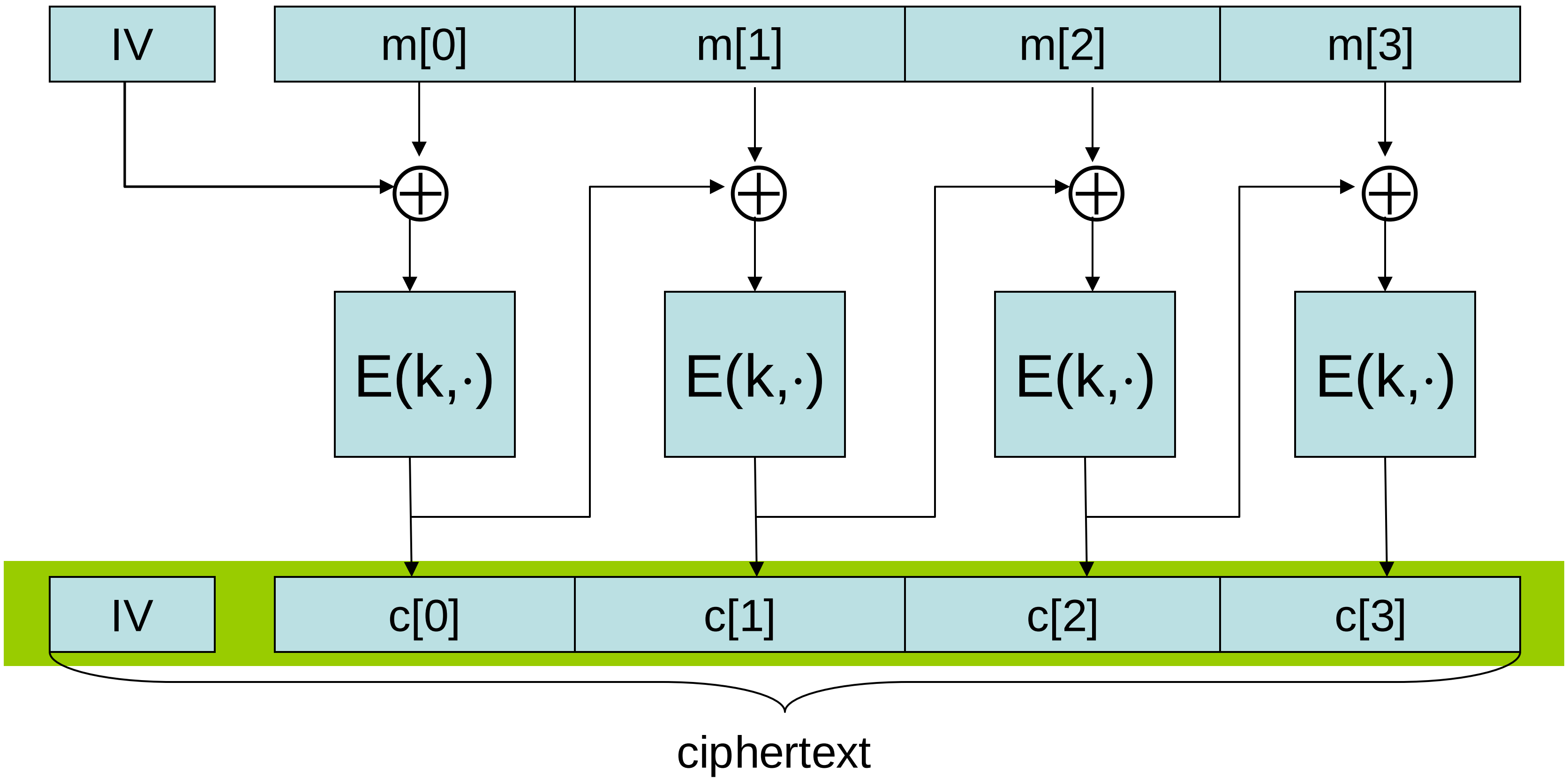
nonce n: a value that changes from msg to msg

(k,n) pair <u>never</u> used more than once

- <u>method 1</u>: encryptor chooses a random nonce, $n \leftarrow \mathcal{N}$
- <u>method 2</u>: nonce is a counter (e.g. packet counter)
  - used when encryptor keeps state from msg to msg
  - if decryptor has same state, need not send nonce with CT

Construction 1:   CBC with random nonce

Cipher block chaining with a <u>random</u> IV   (IV = nonce)



note: CBC where attacker can predict the IV is not CPA-secure. HW.

CBC: CPA Analysis

<u>CBC Theorem</u>: For any L>0,

If E is a secure PRP over (K,X) then

E_{CBC} is a sem. sec. under CPA over (K, X^L, X^{L+1}).

In particular, for a q-query adversary A attacking E_{CBC} there exists a PRP adversary B s.t.:

$$\mathrm{Adv}_{CPA}[A, E_{CBC}] \le 2 \cdot \mathrm{Adv}_{PRP}[B, E] + 2\, q^2\, L^2 / |X|$$

Note: CBC is only secure as long as $q^2 \cdot L^2 \ll |X|$

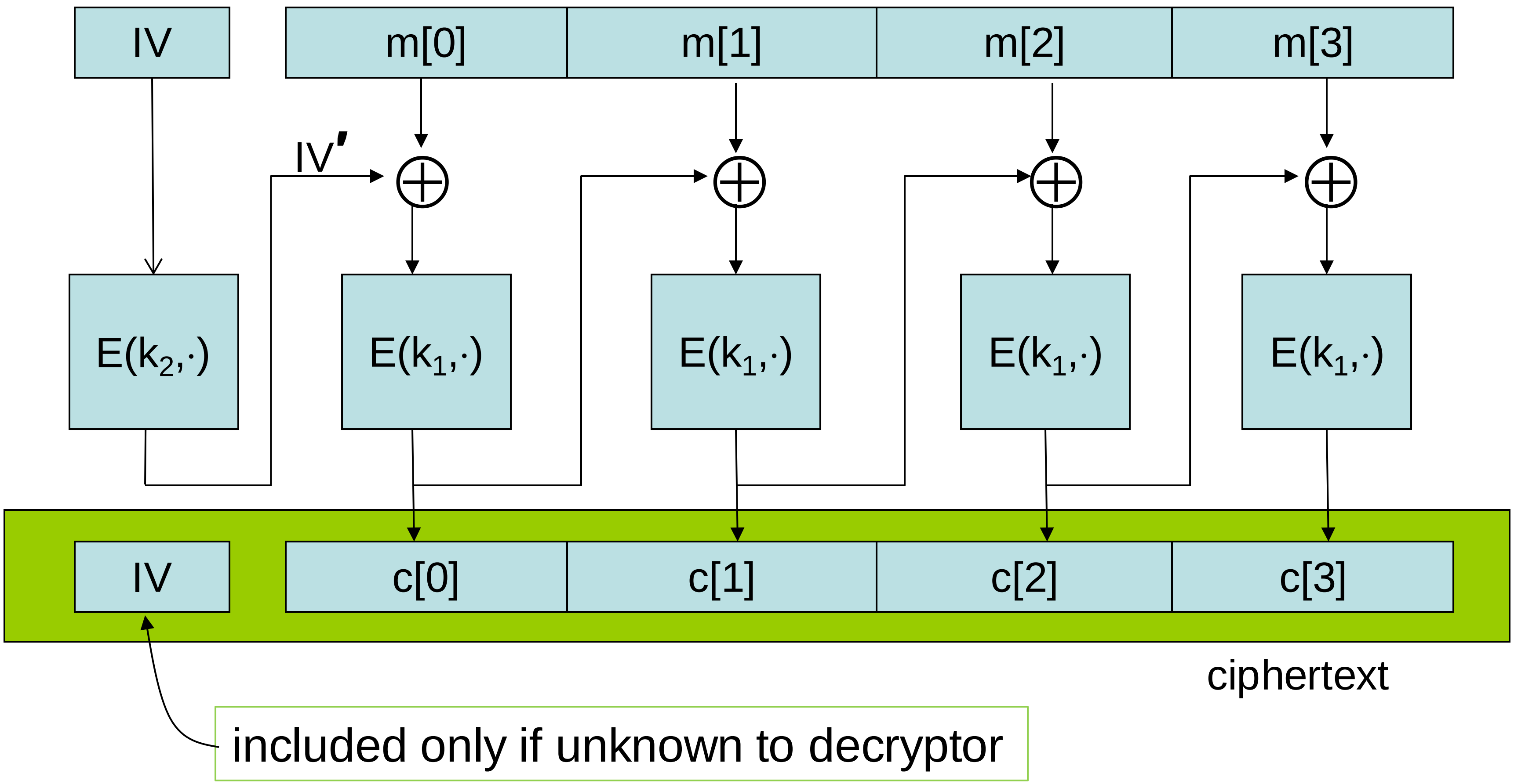
messages enc. with key

max msg length

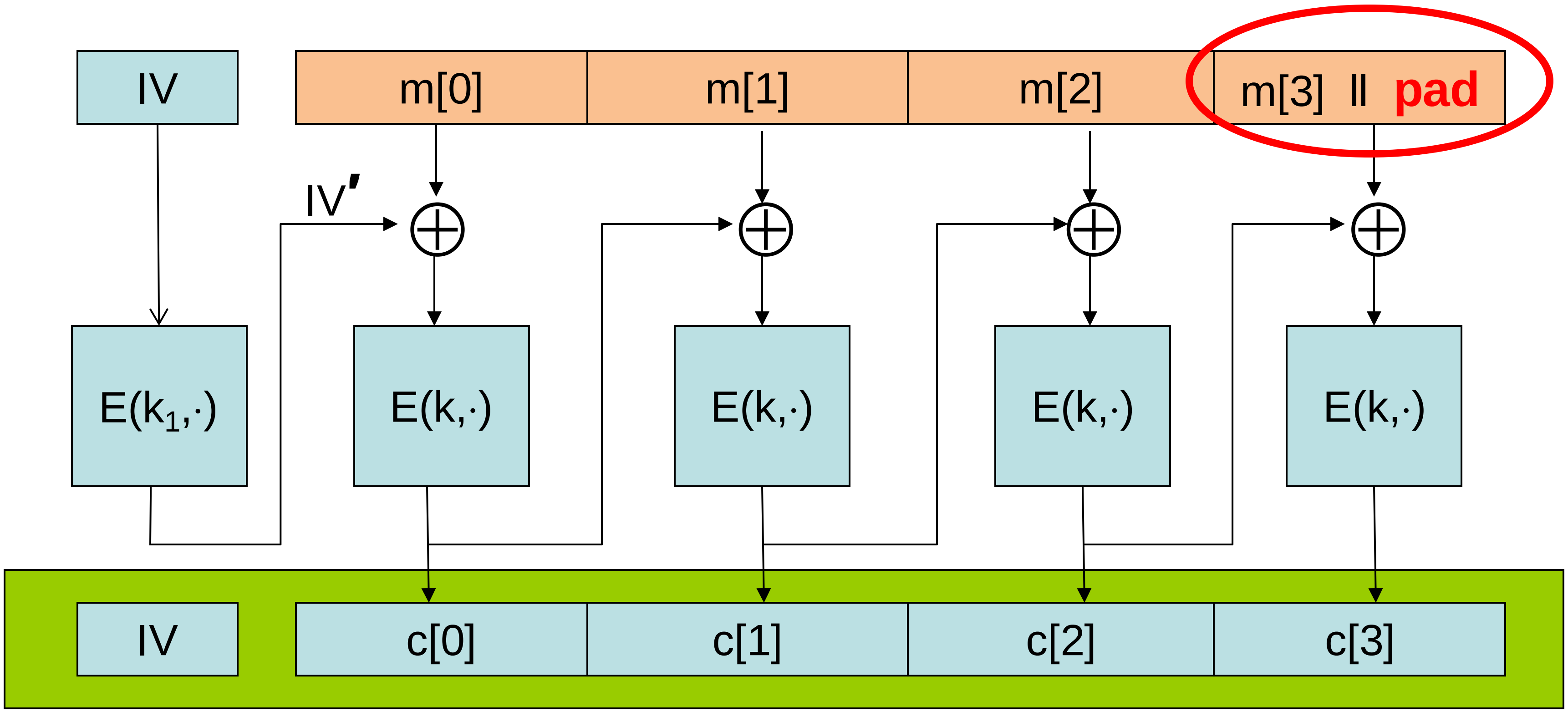
Construction 1': CBC with **unique** nonce

Cipher block chaining with <u>unique</u> IV (IV = nonce)

unique IV means: (key,IV) pair is used for only one message



A CBC technicality: padding



TLS 1.0: if need n-byte pad, n>0, use:

n-1	n-1	•••	n-1

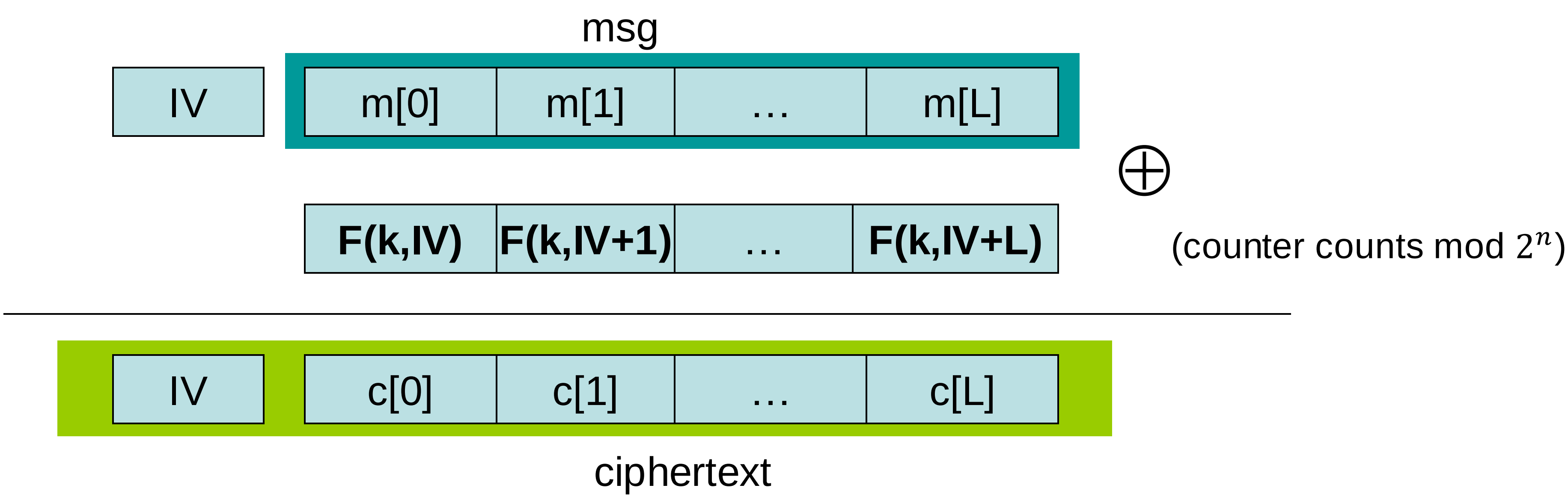
if no pad needed, add a dummy block

pad is removed during decryption

Construction 2: rand ctr-mode

F: PRF defined over (K,X,Y) where X = $\{0,1,\ldots,2^n\text{-}1\}$ and Y = $\{0,1\}^n$

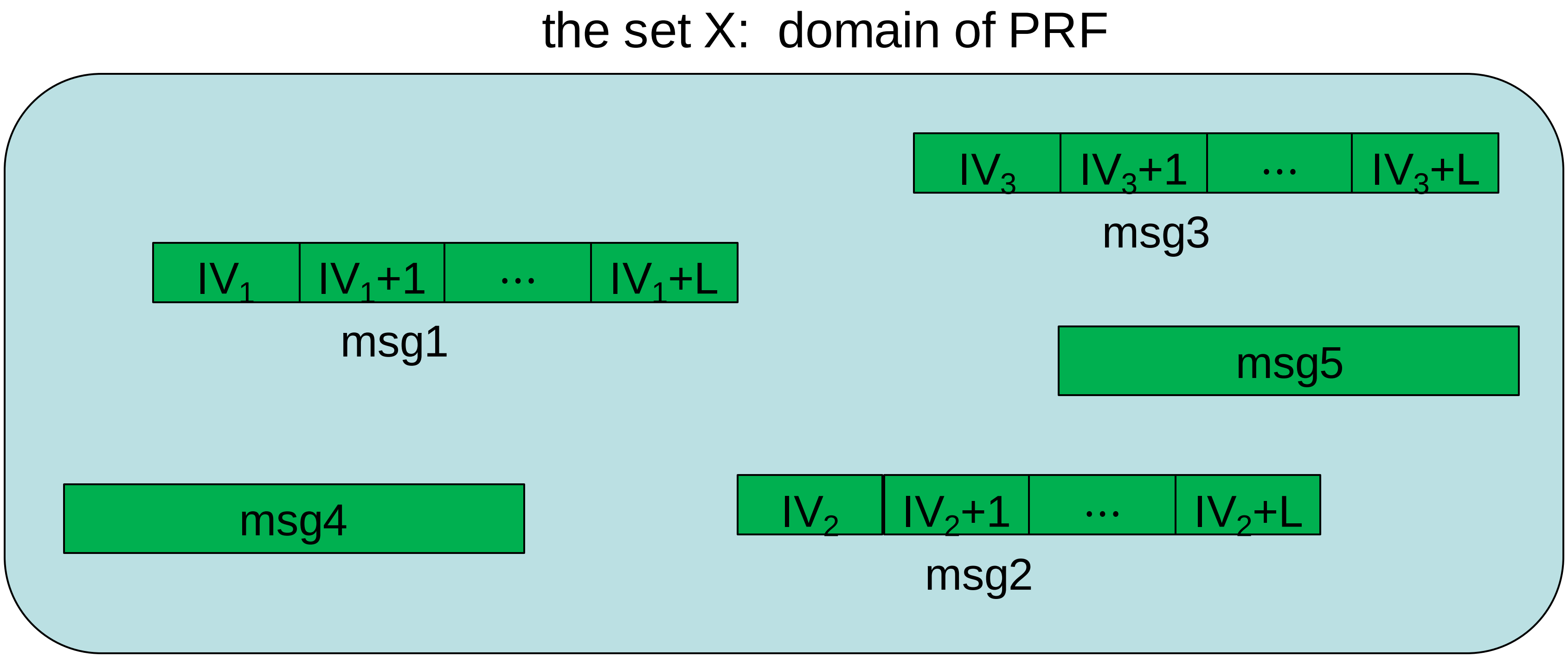
(e.g., n=128)



IV - chosen at random for every message

note: parallelizable (unlike CBC)

Why is this CPA secure?



CPA security holds as long as intervals do not intersect

- q msgs, L blocks each $\Rightarrow$ Pr[intersection] ≤ $\underbrace{2\ q^2\ L\ /\ |X|}_{\text{needs to be negligible}}$

rand ctr-mode: CPA analysis

Randomized counter mode: random IV.

Counter-mode Theorem: For any L>0,

If F is a secure PRF over (K,X,X) then

E_{CTR} is a sem. sec. under CPA over (K, X^L, X^{L+1}).

In particular, for a q-query adversary A attacking E_{CTR} there exists a PRF adversary B s.t.:

$$\mathrm{Adv}_{CPA}[A, E_{CTR}] \leq 2 \cdot \mathrm{Adv}_{PRF}[B, F] + 2\, q^2 L / |X|$$

Note: ctr-mode only secure as long as $q^2 \cdot L \ll |X|$

Better then CBC !

An example

$$\mathrm{Adv}_{\mathrm{CPA}}[A, E_{\mathrm{CTR}}] \leq 2 \cdot \mathrm{Adv}_{\mathrm{PRF}}[B, E] + \mathbf{2\, q^2\, L / |X|}$$

q = # messages encrypted with k , L = length of max msg

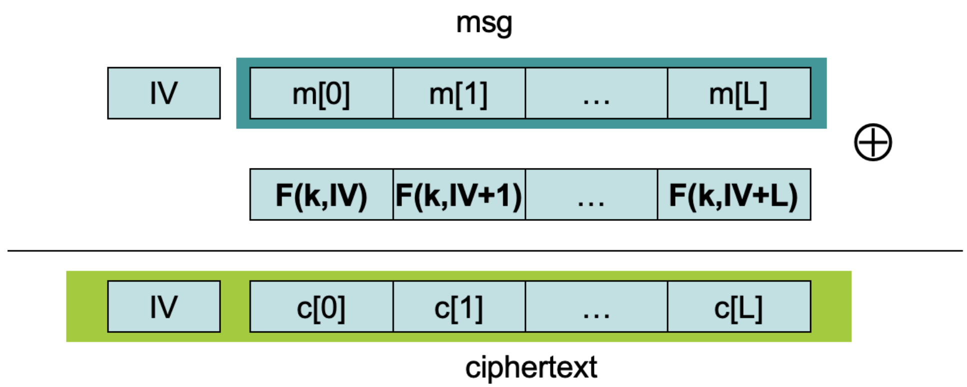
Suppose we want $\mathbf{Adv_{CPA}}[A, E_{\mathrm{CTR}}] \leq \mathbf{1/\, 2^{31}}$

- Then need: $\mathbf{q^2\, L / |X|} \leq \mathbf{1/\, 2^{32}}$
- AES: $|X| = 2^{128} \Rightarrow \mathbf{q}\, \mathbf{L}^{1/2} < 2^{48}$

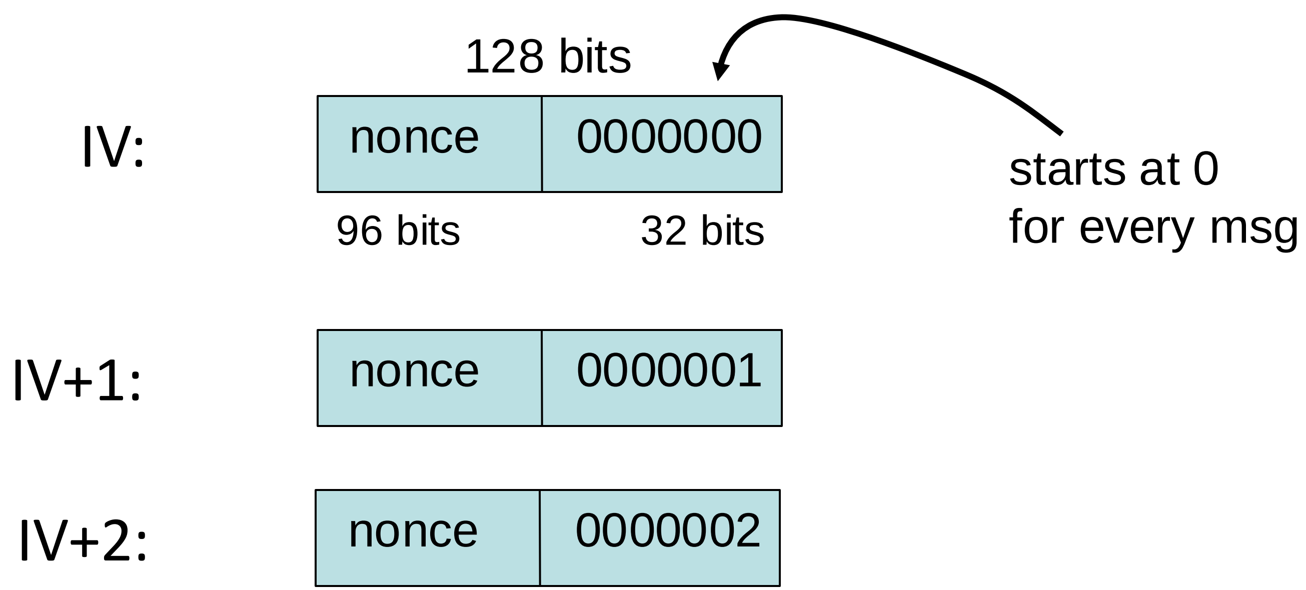
So, after $\underbrace{\mathbf{2^{32}}\ \mathbf{CTs}}$ each of $\underbrace{\mathbf{len}\ \mathbf{2^{32}}}$, must change key

(total of 2^{64} AES blocks)

Construction 2': nonce ctr-mode



To ensure F(k,x) is never used more than once, choose IV as:



Comparison: ctr vs. CBC

	~~CBC~~	**ctr mode**
required primitive	PRP	PRF
parallel processing	No	Yes
security	q^2 L^2 << |X|	q^2 L << |X|
dummy padding block	Yes*	No
1 byte msgs (nonce-based)	16x expansion	no expansion

* for CBC, dummy padding block can be avoided using *ciphertext stealing*

Summary

PRPs and PRFs: a useful abstraction of block ciphers.

We examined two security notions:

1. Semantic security against one-time.
2. Semantic security against many-time CPA.

Note: neither mode ensures data integrity.

Stated security results summarized in the following table:

Power / Goal	**one-time key**	**Many-time key (CPA)**	**CPA and CT integrity**
Sem. Sec.	steam-ciphers det. ctr-mode	rand CBC rand ctr-mode	later

Attacks on block ciphers

Goal: distinguish block cipher from a random permutation

- if this can be done efficiently then block cipher is broken

Harder goal:

find key k given many $c_i = E(k, m_i)$ for random m_i

(1) Linear and differential attacks

[BS'89,M'93]

Given *many* (m_i, c_i) pairs, can recover key much faster than exhaustive search

<u>Linear cryptanalysis</u> (overview) : let c = DES(k, m)

Suppose for random k, m :

$$\Pr\Big[\ m[i_1]\oplus\cdots\oplus m[i_r]\ \bigoplus\ c[j_j]\oplus\cdots\oplus c[j_v]\ =\ k[l_1]\oplus\cdots\oplus k[l_u]\ \Big] = ½ + \varepsilon$$

For some ε.

For DES, this exists with $\varepsilon = 1/2^{21} \approx 0.0000000477$!!

Linear attacks

$$\Pr\Big[\, m[i_1] \oplus \cdots \oplus m[i_r] \;\bigoplus\; c[j_j] \oplus \cdots \oplus c[j_v] \;=\; k[l_1] \oplus \cdots \oplus k[l_u] \,\Big] = ½ + \varepsilon$$

<u>Thm</u>: given $1/\varepsilon^2$ random pairs $\big(m,\ c = DES(k, m)\big)$ then

$$k[l_1] \oplus \ldots \oplus k[l_u] \;=\; \mathrm{MAJ}\Big[\, m[i_1] \oplus \ldots \oplus m[i_r] \bigoplus c[j_j] \oplus \ldots \oplus c[j_v] \,\Big]$$

with prob. ≥ 97.7%

⇒ with $1/\varepsilon^2$ inp/out pairs can find $k[l_1] \oplus \ldots \oplus k[l_u]$ in time $\approx 1/\varepsilon^2$

Linear attacks

For DES, $\varepsilon = 1/2^{21} \Rightarrow$

with 2^{42} inp/out pairs can find $k[l_1] \oplus \ldots \oplus k[l_u]$ in time 2^{42}

Roughly speaking: can find 14 key "bits" this way in time 2^{42}

Brute force remaining $56-14=42$ bits in time 2^{42}

Attack time: $\approx 2^{43}$ ($\ll 2^{56}$) with 2^{42} random inp/out pairs

Lesson

A tiny bit of linearly leads to a 2^{42} time attack.

$\Rightarrow$ don’t design ciphers yourself !!

(2) Side channel attacks on software AES

Attacker measures the **<u>time</u>** to compute AES128(k,m) for many random blocks m.

- Suppose that the 256-byte S table is not in L1 cache at the start of each invocation

  $\Longrightarrow$ time to encrypt reveals the order in which S entries are accessed

  $\Longrightarrow$ leaks info. that can compromise entire key

Lesson: don't implement AES yourself !

Mitigation: AES-NI or use vetted software (e.g., BoringSSL)

(3) Quantum attacks

Generic search problem:

Let $f: X \longrightarrow \{0,1\}$ be a function.

Goal: find $x \in X$ s.t. $f(x) = 1$.

Classical computer: best generic algorithm time = $O(|X|)$

Quantum computer [Grover '96] : time = $O(|X|^{1/2})$

(requires a long running quantum computation)

Quantum exhaustive search

Given m, c=E(k,m) define

$$f(k) = \begin{cases} 1 & \text{if } E(k,m) = c \\ 0 & \text{otherwise} \end{cases}$$

Grover $\Rightarrow$ quantum computer can find k in time $O(|K|^{1/2})$

AES128: quantum key recovery time $\approx 2^{64}$

Adversary has access to a quantum computer $\Longrightarrow$

encrypt data using a cipher with 256-bit keys (AES256)

THE END