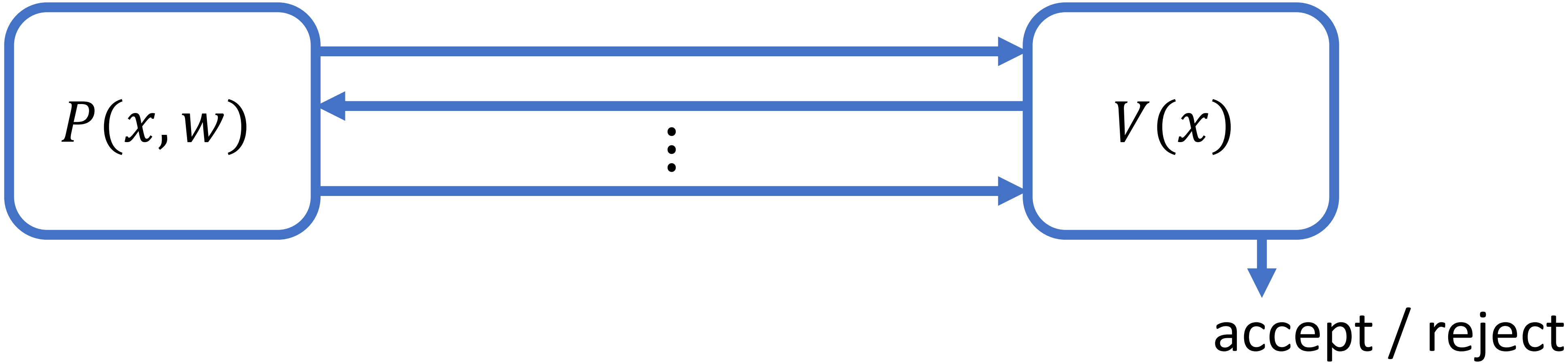
Quasi-Optimal SNARGs via Linear Multi-Prover Interactive Proofs

Dan Boneh, Yuval Ishai, Amit Sahai, and David J. Wu

Interactive Arguments for NP

$$\mathcal{L}_C = \{x : C(x, w) = 1 \text{ for some } w\}$$



Completeness: $C(x, w) = 1 \Longrightarrow \Pr[\langle P(x, w), V(x) \rangle = 1] = 1$

Soundness: for all provers $P^\star$ of size 2^λ:

$$x \notin \mathcal{L}_C \Longrightarrow \Pr[\langle P^\star(x), V(x) \rangle = 1] \leq 2^{-\lambda}$$

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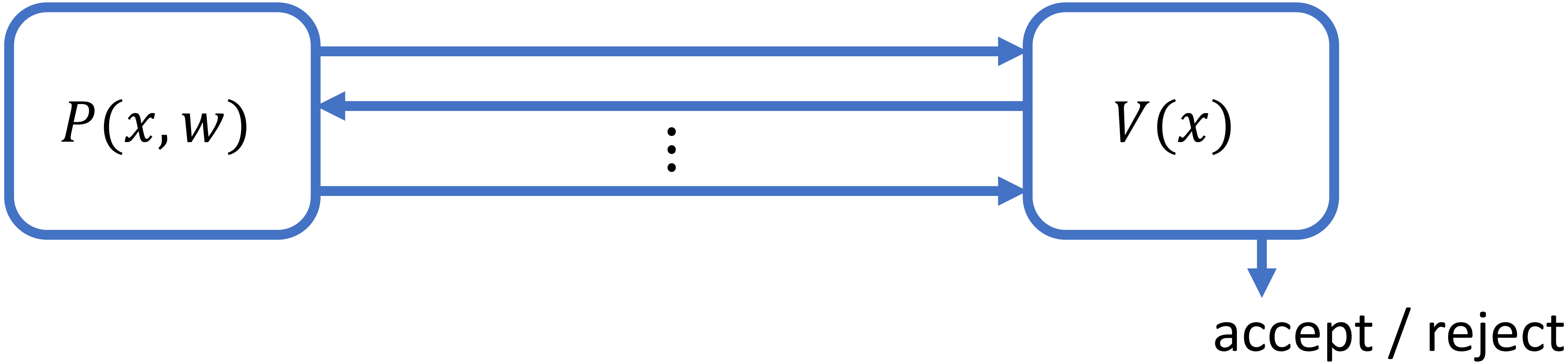

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λ is a security parameter

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Succinct Arguments

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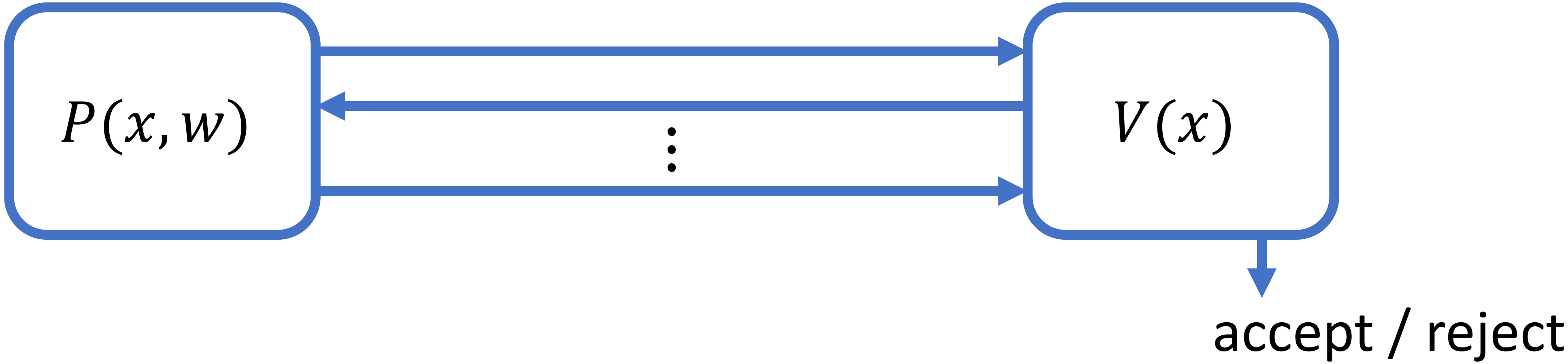


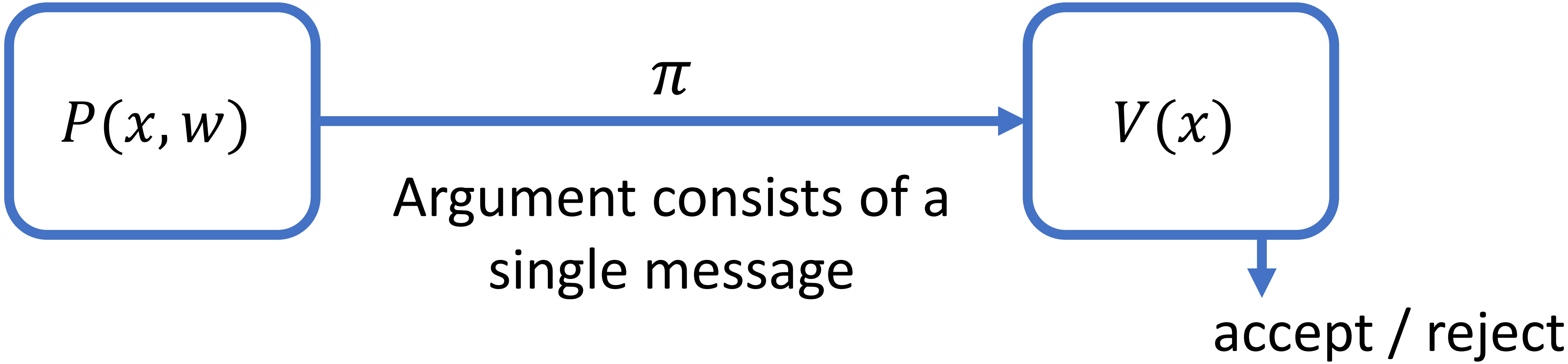
Argument system is *succinct* if:

- Prover communication is $\text{poly}(\lambda + \log|C|)$
- V can be implemented by a circuit of size $\text{poly}(\lambda + |x| + \log|C|)$

Verifier complexity significantly smaller than classic NP verifier

Succinct Non-Interactive Arguments (SNARGs)

Instantiation: "CS proofs" in the random oracle model [Mic94]



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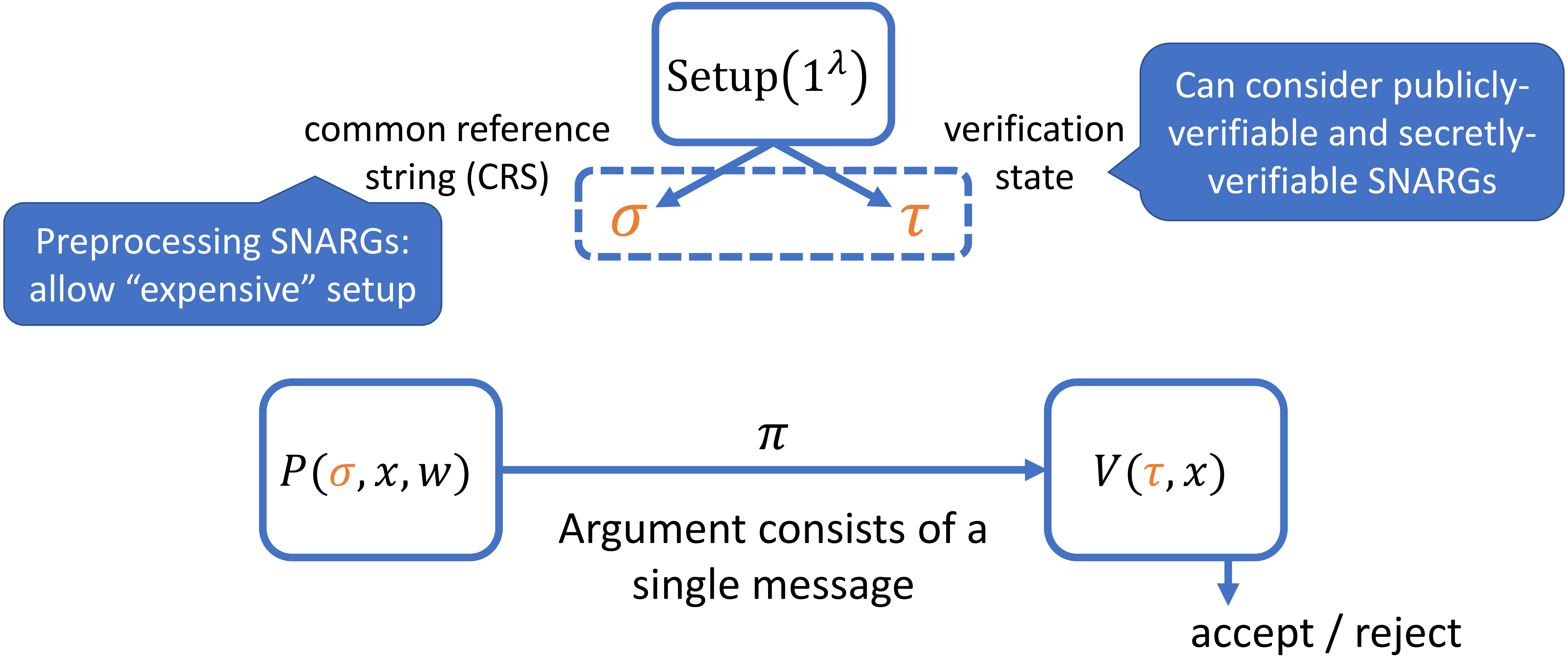

Complexity Metrics for SNARGs

Soundness: for all provers $P^\star$ of size 2^λ:

$$x \notin \mathcal{L}_C \Longrightarrow \Pr[\langle P^\star(x), V(x)\rangle = 1] \leq 2^{-\lambda}$$

How short can the proofs be?

$$|\pi| = \Omega(\lambda)$$

Even in the designated-verifier setting

[See paper for details]

How much work is needed to generate the proof?

$$|P| = \Omega(|C|)$$

Quasi-Optimal SNARGs

Soundness: for all provers $P^\star$ of size 2^λ:

$$x \notin \mathcal{L}_C \Longrightarrow \Pr[\langle P^\star(x), V(x)\rangle = 1] \le 2^{-\lambda}$$

A SNARG (for Boolean circuit satisfiability) is <u>quasi-optimal</u> if it satisfies the following properties:

- Quasi-optimal succinctness:

$$|\pi| = \lambda \cdot \mathrm{polylog}(\lambda, |C|) = \tilde{O}(\lambda)$$

- Quasi-optimal prover complexity:

$$|P| = \tilde{O}(|C|) + \mathrm{poly}(\lambda, \log|C|)$$

Quasi-Optimal SNARGs

Construction	Prover Complexity	Proof Size	Assumption
CS Proofs [Mic94]	$\tilde{O}(\|C\|)$	$\tilde{O}(\lambda^2)$	Random Oracle
Groth [Gro16]	$\tilde{O}(\lambda\|C\|)$	$\tilde{O}(\lambda)$	Generic Group
Groth [Gro10]	$\tilde{O}(\lambda\|C\|^2 + \|C\|\lambda^2)$	$\tilde{O}(\lambda)$	Knowledge of Exponent
GGPR [GGPR12]	$\tilde{O}(\lambda\|C\|)$	$\tilde{O}(\lambda)$	
BCIOP (Pairing) [BCIOP13]	$\tilde{O}(\lambda\|C\|)$	$\tilde{O}(\lambda)$	Linear-Only Encryption
BISW (LWE/RLWE) [BISW17]	$\tilde{O}(\lambda\|C\|)$	$\tilde{O}(\lambda)$	Linear-Only Vector Encryption

For simplicity, we ignore low order terms $\mathrm{poly}(\lambda, \log|C|)$

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This work	$\tilde{O}(\|C\|)$	$\tilde{O}(\lambda)$	Linear-Only Vector Encryption

This Work

New framework for building preprocessing SNARGs (following [BCIOP13, BISW17])

Step 1 (information-theoretic):

- Linear multi-prover interactive proofs (linear MIPs)
- This work: first construction of a quasi-optimal linear MIP

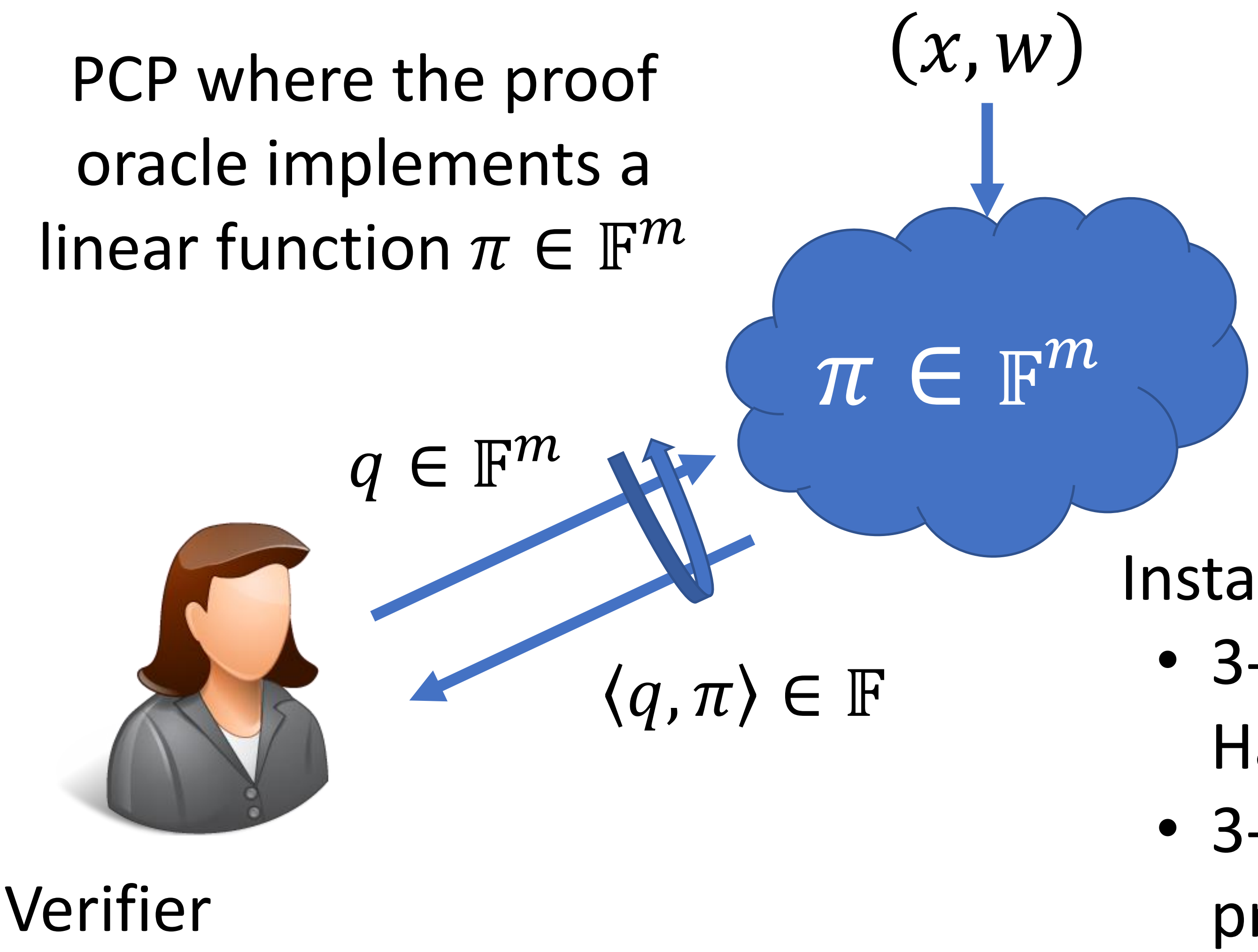
Step 2 (cryptographic):

- Linear-only vector encryption to simulate linear MIP model
- This work: linear MIP $\Longrightarrow$ preprocessing SNARG

Results yield the first quasi-optimal SNARG (from linear-only vector encryption over rings)

Linear PCPs [IKO07]

PCP where the proof oracle implements a linear function $\pi \in \mathbb{F}^m$



In these instantiations, verifier is <u>oblivious</u> (queries independent of statement)

Instantiations:

- 3-query LPCP based on the Walsh-Hadamard code: $m = O(|C|^2)$ [ALMSS92]
- 3-query LPCP based on quadratic span programs: $m = O(|C|)$ [GGPR13]

From Linear PCPs to SNARGs [BCIOP13]

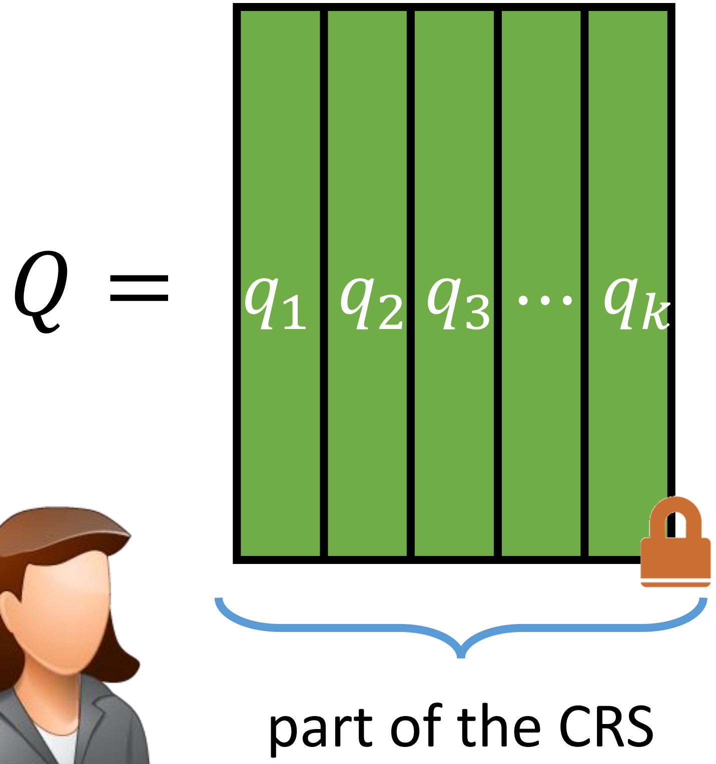
Verifier encrypts its queries using a <u>linear-only</u> encryption scheme

$$Q = \underbrace{\begin{bmatrix} q_1 & q_2 & q_3 & \cdots & q_k \end{bmatrix}}_{\text{part of the CRS}}$$

CPs to SNARGs [BCIOP13]

Encryption scheme that <u>only</u> supports linear homomorphism

Verifier encrypts its queries using a <u>linear-only</u> encryption scheme



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Prover constructs linear PCP π from (x, w)

$$(x, w) \rightarrow \pi \in \mathbb{F}^m$$

Prover homomorphically computes responses to linear PCP queries

$\langle \pi, q_1 \rangle$	$\langle \pi, q_2 \rangle$	$\cdots$	$\langle \pi, q_k \rangle$

SNARG proof

From Linear PCPs to SNARGs [BCIOP13]

Evaluating inner product requires $O(km)$ homomorphic operations on ciphertexts: prover complexity $O(\lambda) \cdot O(km) = O(\lambda|C|)$

$$Q = \begin{array}{|c|c|c|c|c|} q_1 & q_2 & q_3 & \cdots & q_k \end{array}$$

Proof consists of a <u>constant</u> number of ciphertexts: total length $O(\lambda)$ bits

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Prover constructs linear PCP π from (x, w)

(x, w)

We pay $O(\lambda)$ for each homomorphic operation. Can we reduce this?

Prove ... es
responses ... eries

$\langle \pi, q_1 \rangle$	$\langle \pi, q_2 \rangle$	$\cdots$	$\langle \pi, q_k \rangle$

SNARG proof

Linear-Only Encryption over Rings

Consider encryption scheme over a polynomial ring $R_p = \mathbb{Z}_p[x]/\Phi_\ell(x) \cong \mathbb{F}_p^\ell$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_\ell \end{pmatrix} + \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_\ell' \end{pmatrix} = \begin{pmatrix} x_1 + {x_1}' \\ x_2 + x_2' \\ x_3 + x_3' \\ \vdots \\ x_\ell + x_\ell' \end{pmatrix}$$

Homomorphic operations correspond to <u>component-wise</u> additions and scalar multiplications

Plaintext space can be viewed as a vector of field elements

Using RLWE-based encryption schemes, can encrypt $\ell = \tilde{O}(\lambda)$ field elements ($p = \mathrm{poly}(\lambda)$) with ciphertexts of size $\tilde{O}(\lambda)$

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Homomorphic operations

Amortized cost of homomorphic operation on a single field element is $\mathrm{polylog}(\lambda)$

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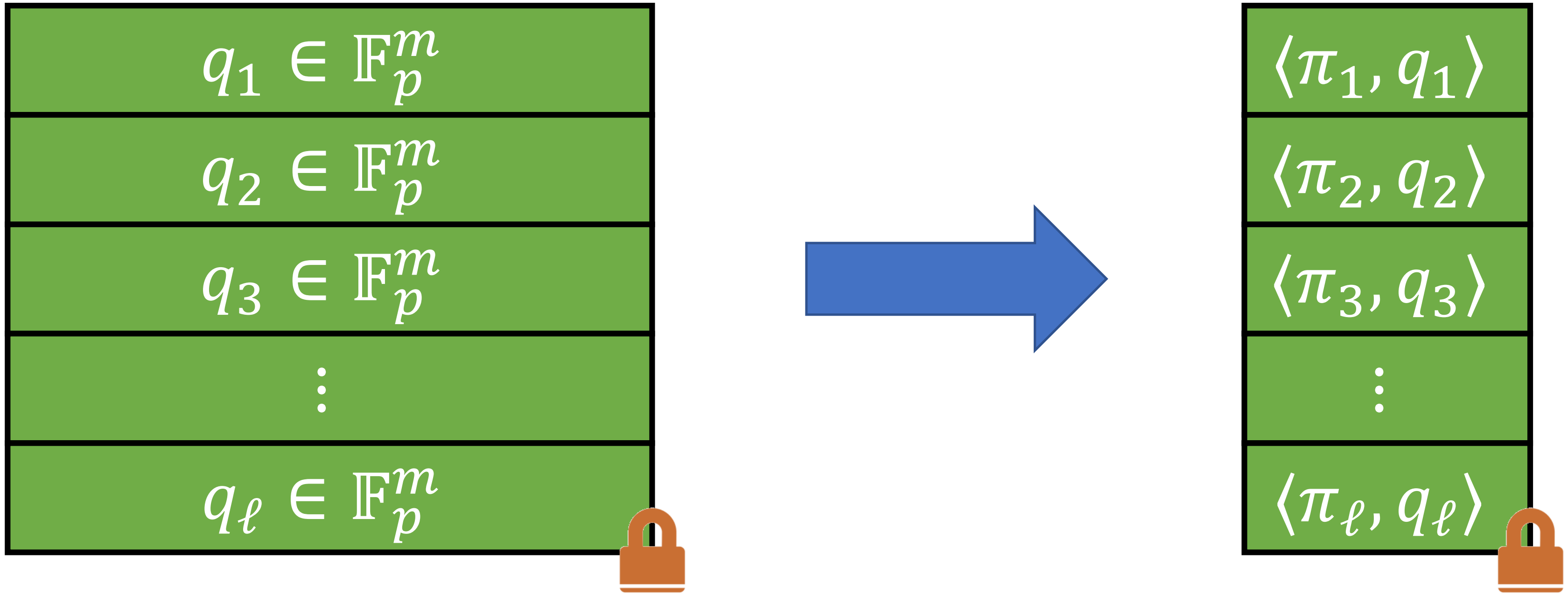
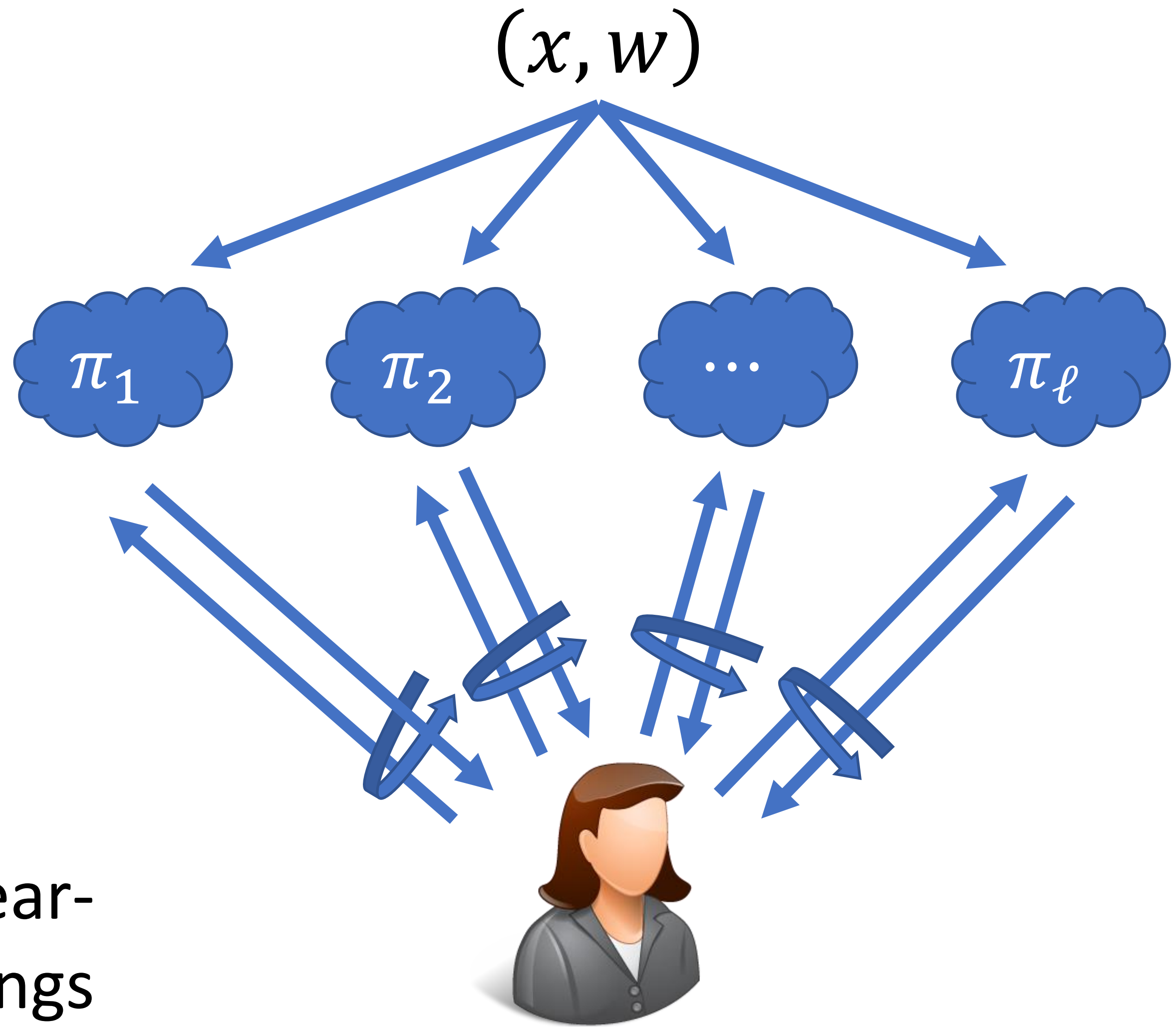
Linear-Only Encryption over Rings



Given encrypted set of query vectors, prover can homomorphically apply <u>independent</u> linear functions to each slot

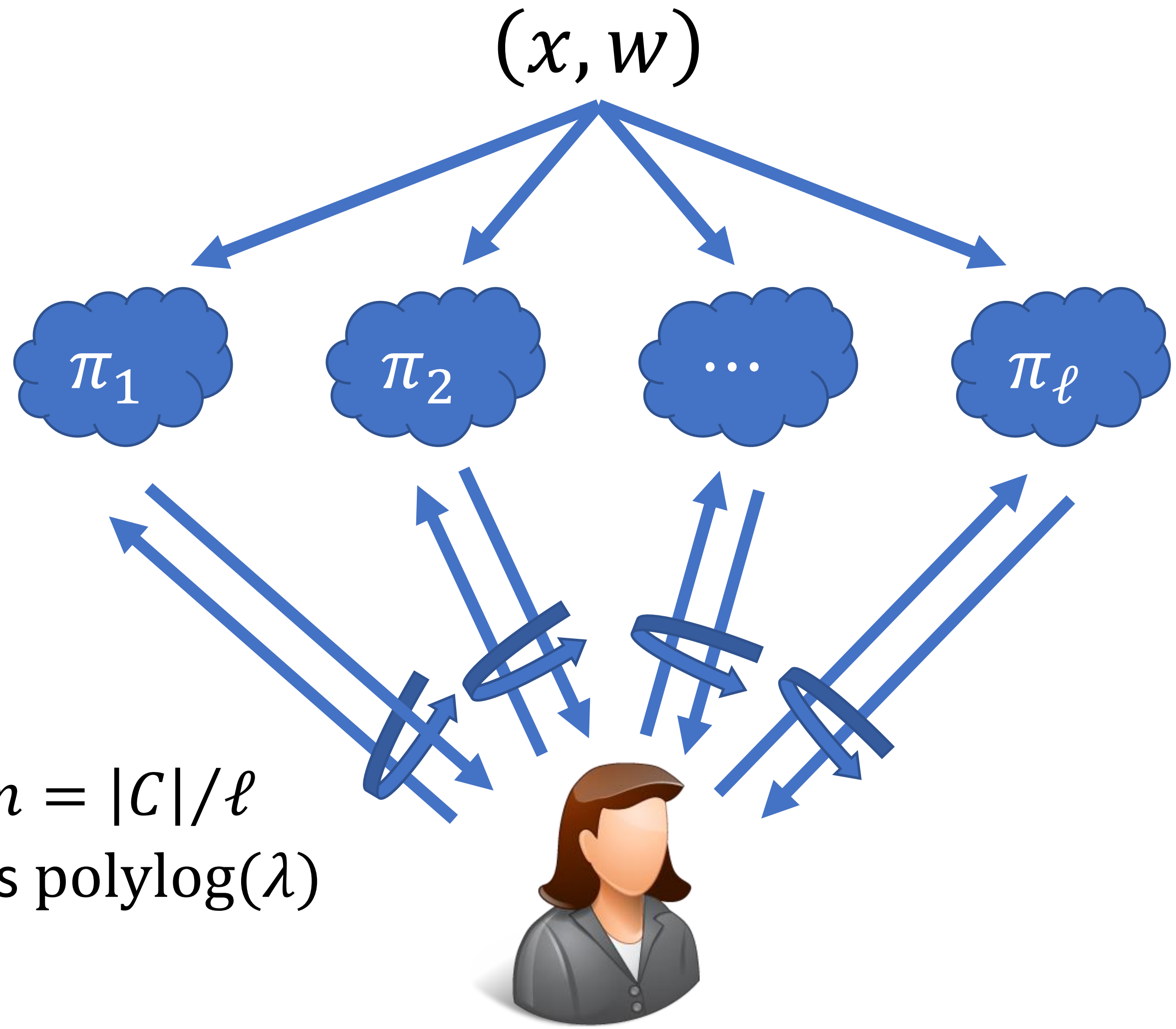
Linear Multi-Prover Interactive Proofs (MIPs)



Verifier has oracle access to multiple linear proof oracles

Can convert linear MIP to preprocessing SNARG using linear-only (vector) encryption over rings

Linear Multi-Prover Interactive Proofs (MIPs)



Suppose

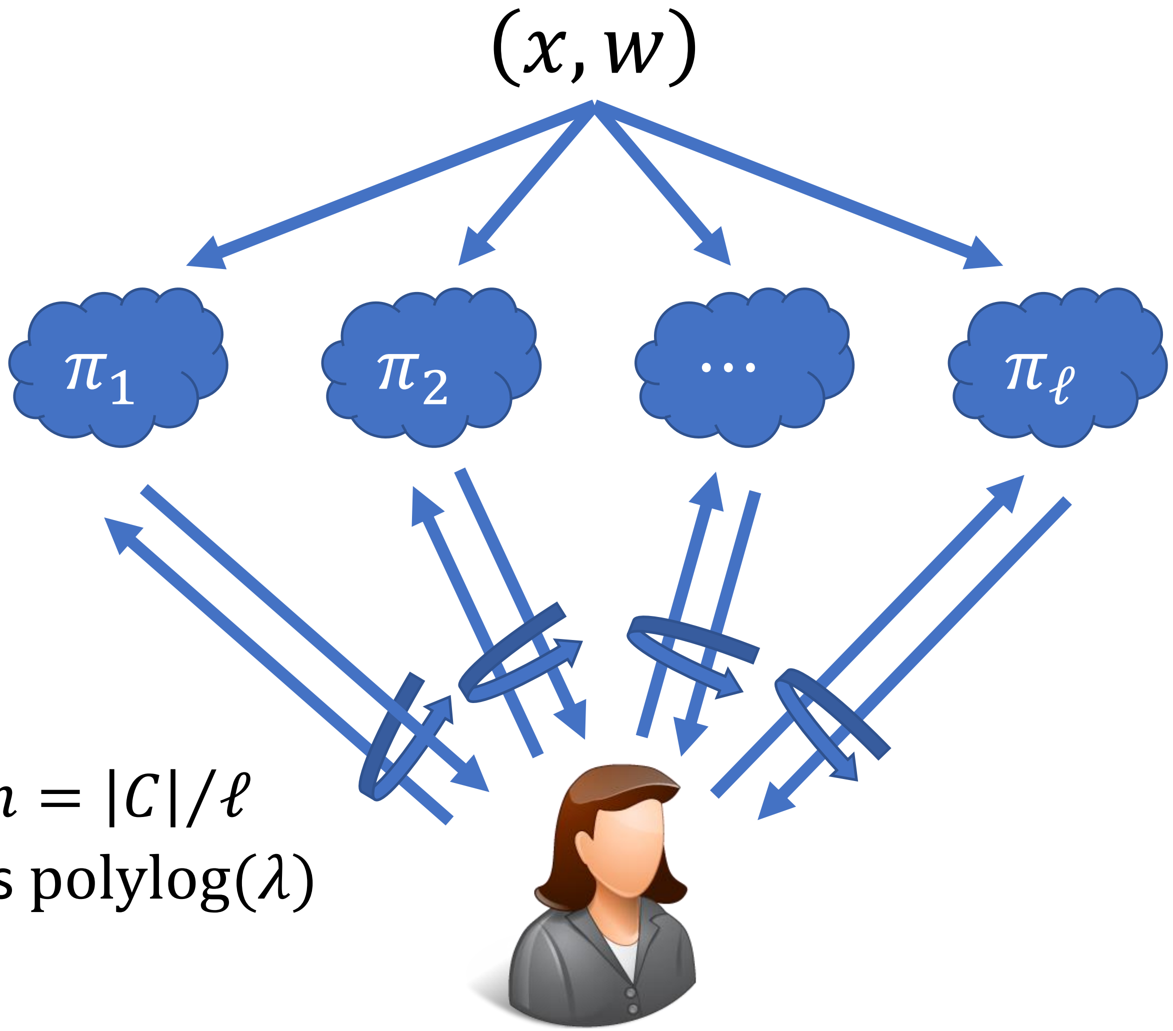
- Number of provers $\ell = \tilde{O}(\lambda)$
- Proofs $\pi_1, \dots, \pi_\ell \in \mathbb{F}_p^m$ where $m = |C|/\ell$
- Number of queries to each π_i is $\text{polylog}(\lambda)$

Then, linear MIP is quasi-optimal

Linear Multi-Prover Interactive Proofs (MIPs)

Prover complexity:
$\tilde{O}(\ell m) = \tilde{O}(|C|)$

Linear MIP size:
$O\big(\ell \cdot \mathrm{polylog}(\lambda)\big) = \tilde{O}(\lambda)$



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Linear Multi-Prover Interactive Proofs (MIPs)

Goal: Construct quasi-optimal linear MIP (with soundness $2^{-\lambda}$) and following properties:

- Number of provers is $\tilde{O}(\lambda)$
- Each proof has length $\tilde{O}(|C|/\lambda)$
- Proofs are over a polynomial-size field: $p = \text{poly}(\lambda)$
- Query complexity is $\text{polylog}(\lambda)$

More provers, shorter (individual) proofs

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Linear PCPs used in [BCIOP13] require a field of size $2^{\Omega(\lambda)}$

Can we use existing linear PCPs?

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Linear PCPs used in [BISW17] have query complexity $\Omega(\lambda)$

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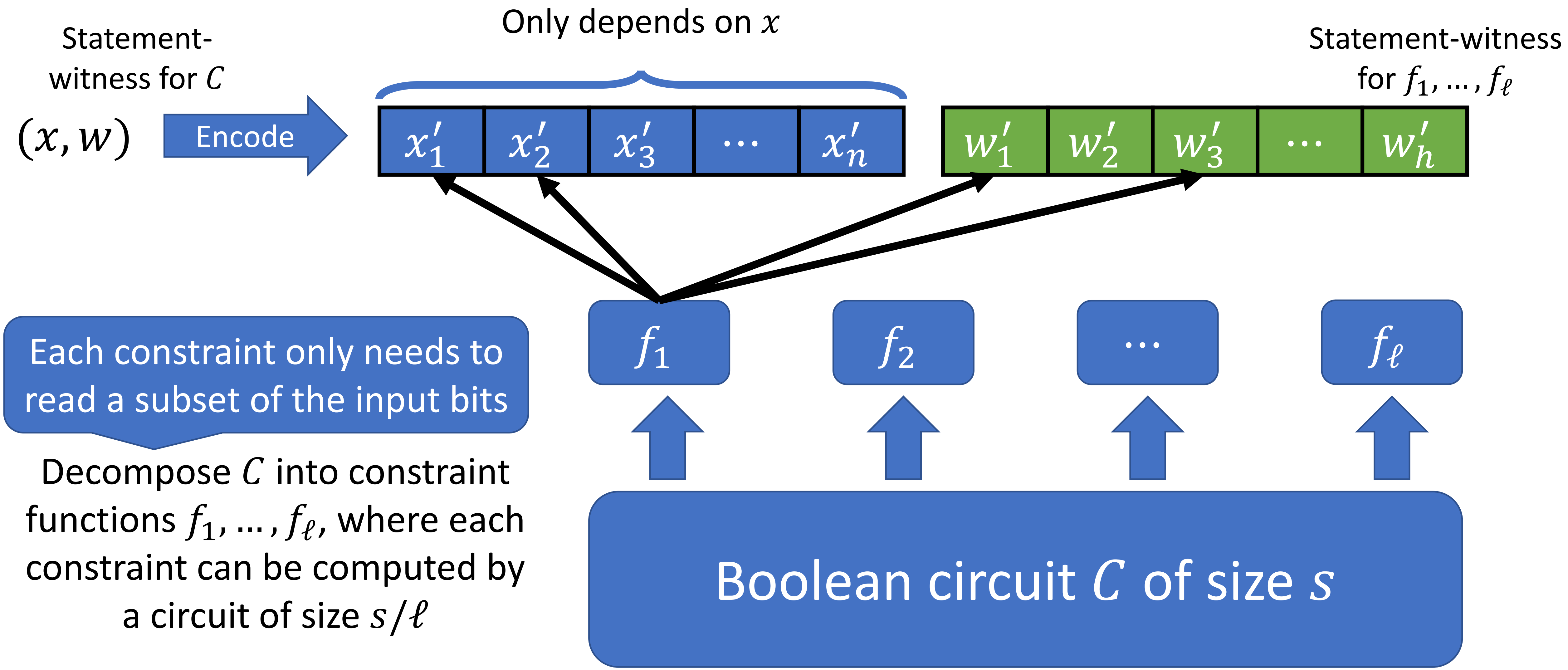
This work: Construction of a quasi-optimal linear MIP for Boolean circuit satisfiability

Quasi-Optimal Linear MIPs

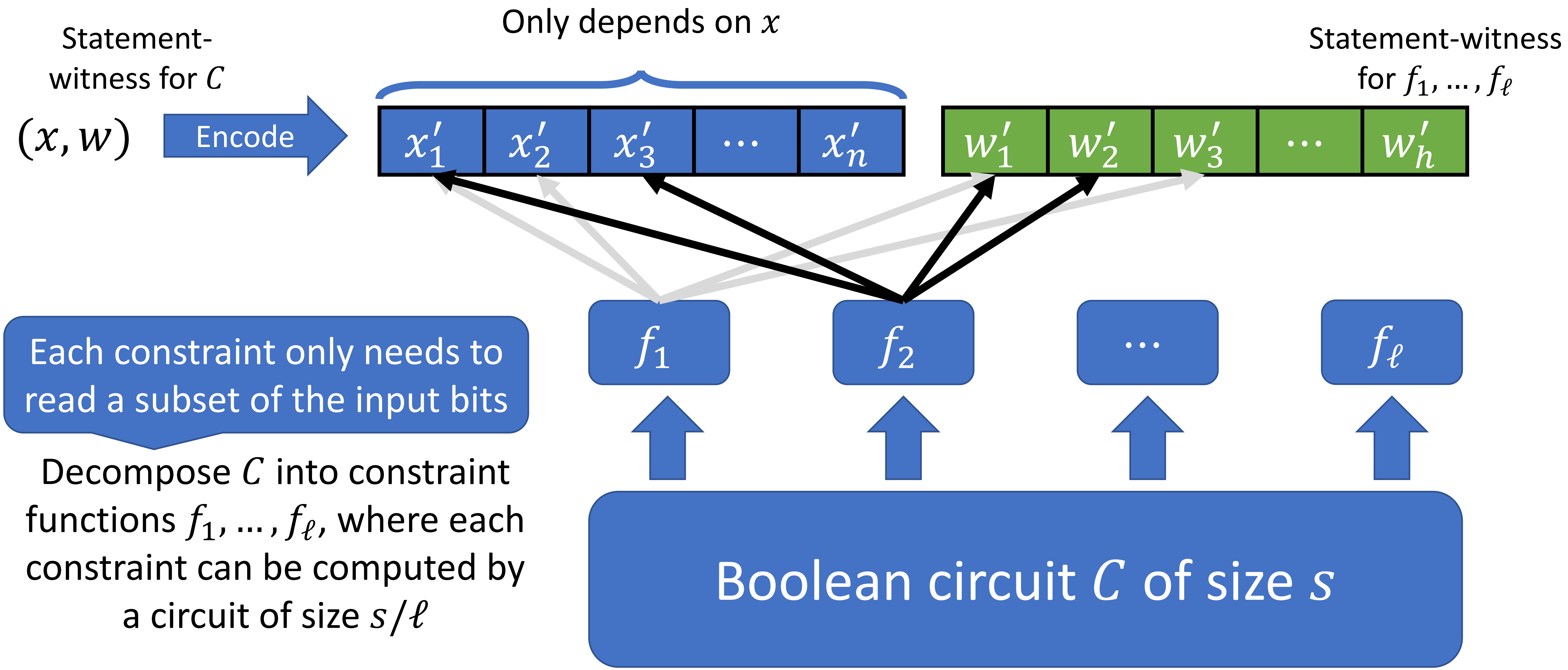
This work: Construction of a quasi-optimal linear MIP for Boolean circuit satisfiability

Robust Decomposition



Decompose C into constraint functions $f_1, \ldots, f_\ell$, where each constraint can be computed by a circuit of size s/ℓ

Robust Decomposition



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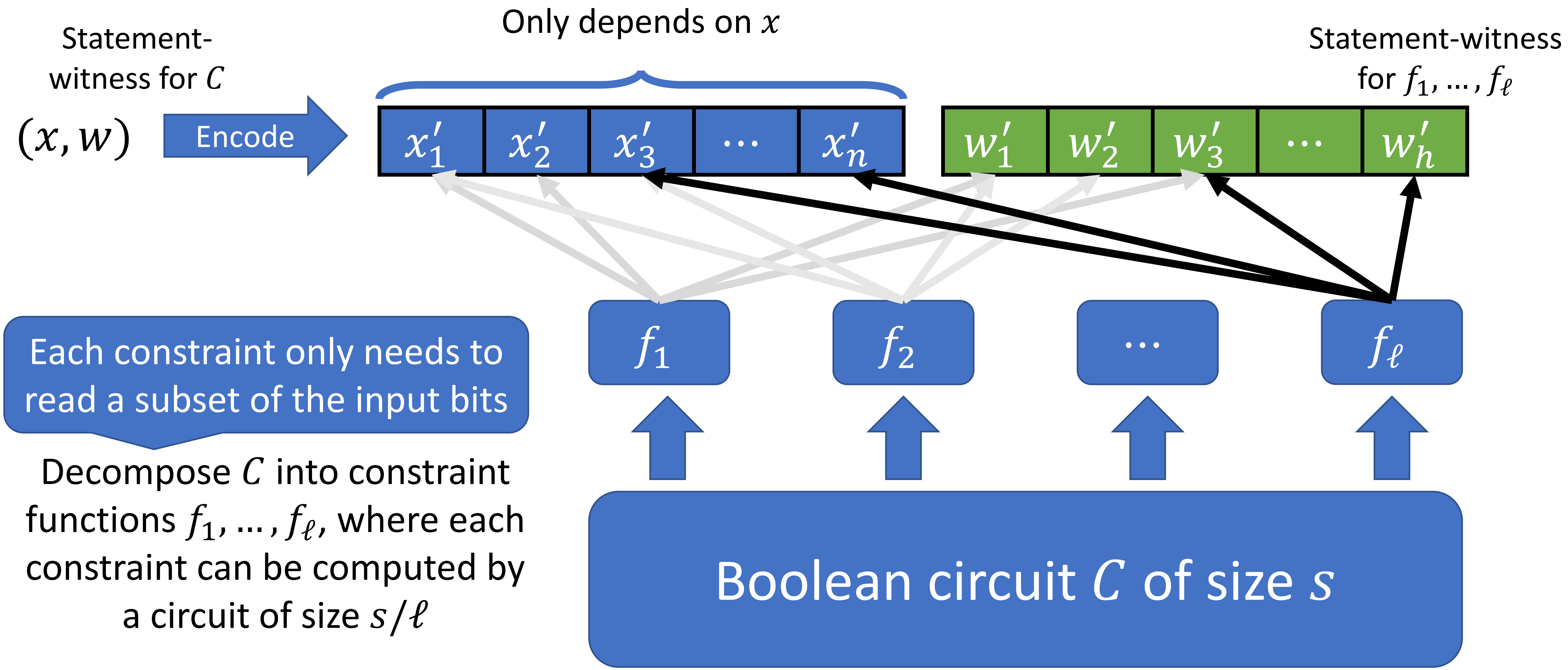

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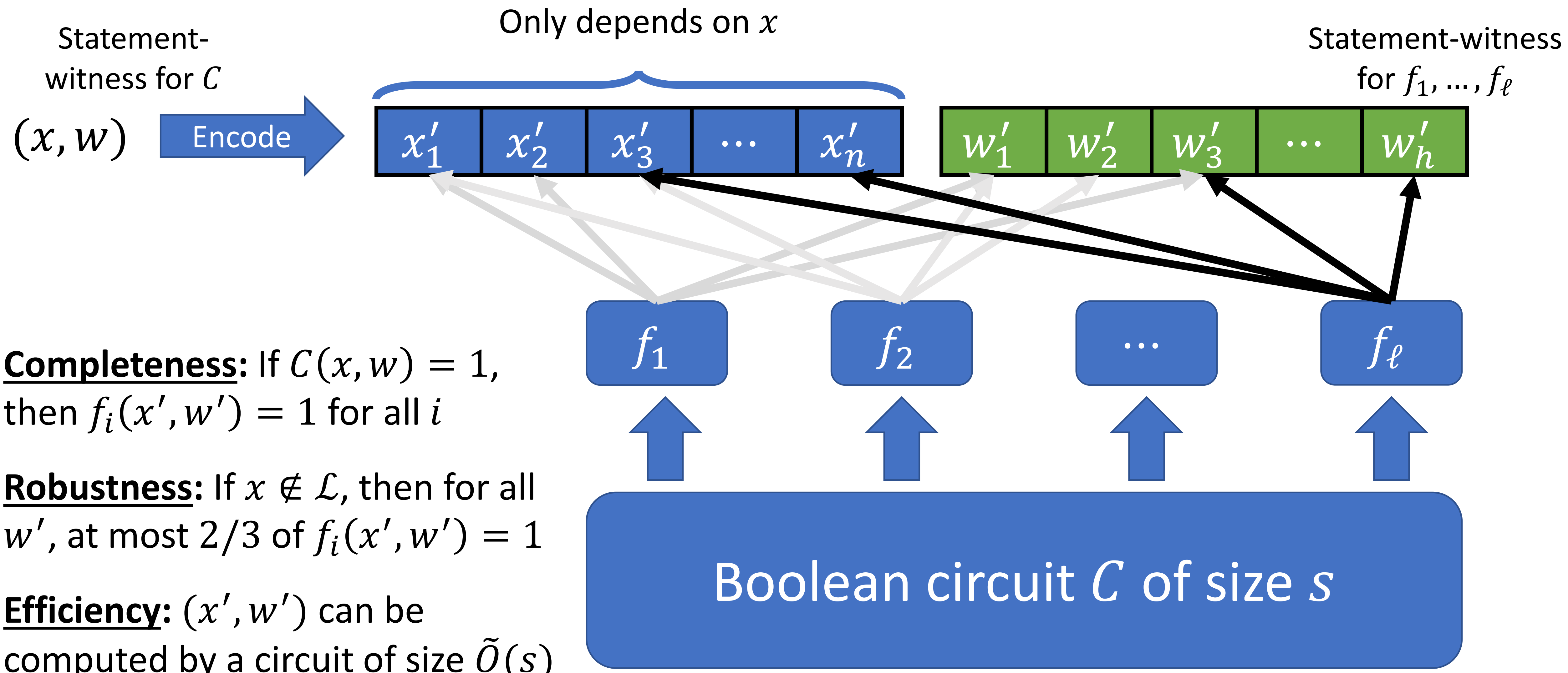


Completeness: If $C(x, w) = 1$, then $f_i(x', w') = 1$ for all i

Robustness: If $x \notin \mathcal{L}$, then for all w', at most $2/3$ of $f_i(x', w') = 1$

Efficiency: (x', w') can be computed by a circuit of size $\tilde{O}(s)$

Robust Decomposition

Boolean circuit C of size s

f_1 — π_1

f_2 — π_2

$\vdots$ — $\vdots$

f_ℓ — π_ℓ

$(x, w) \xrightarrow{\text{Encode}} (x', w')$

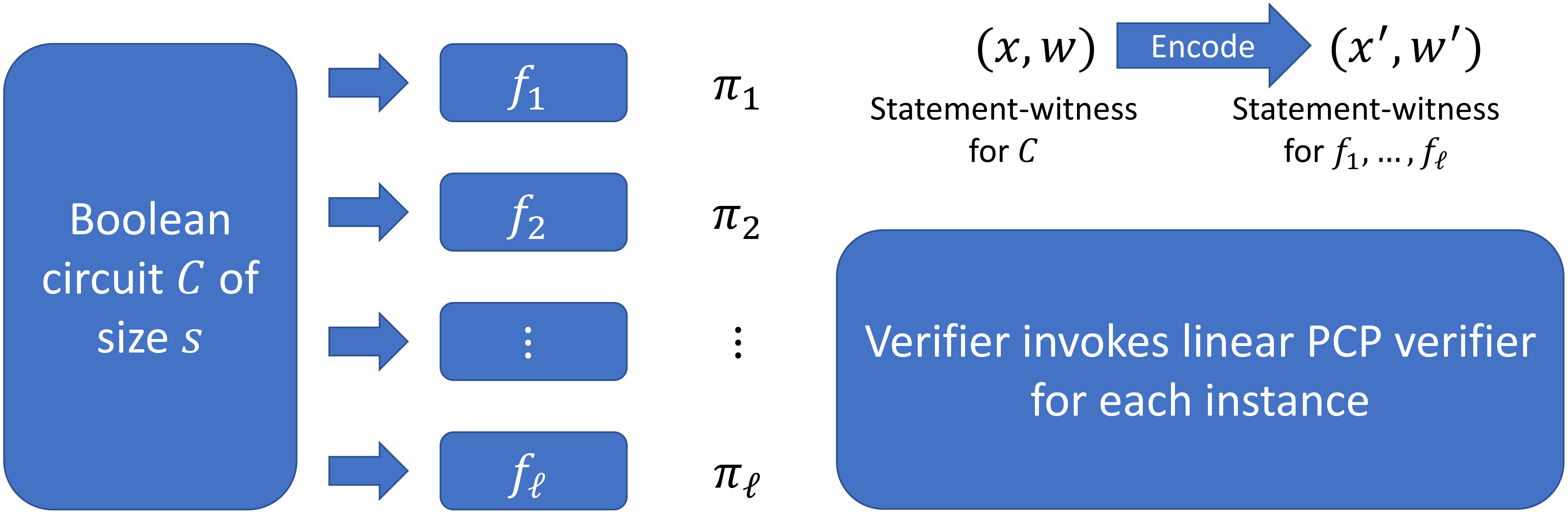
Statement-witness for C

Statement-witness for $f_1, \ldots, f_\ell$

Using constant-query linear PCP based on QSPs [GGPR13], $\pi_i \in \mathbb{F}_p^m$ where $m = O(|C|/\ell)$ and provides soundness $1/\mathrm{poly}(\lambda)$

π_i: linear PCP that $f_i(x', \cdot)$ is satisfiable
(instantiated over $\mathbb{F}_p$ where $p = \mathrm{poly}(\lambda)$)

Robust Decomposition



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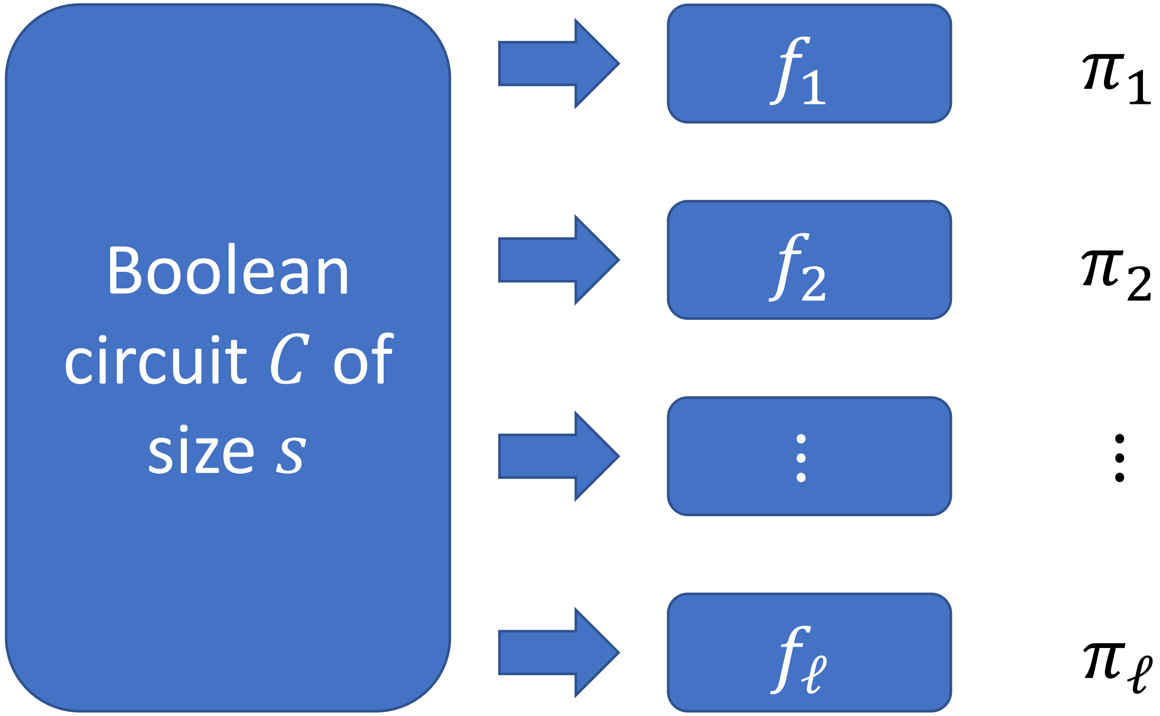

Completeness: Follows by completeness of decomposition and linear PCPs

Soundness: Each linear PCP provides $1/\text{poly}(\lambda)$ soundness and for false statement, at least $1/3$ of the statements are false, so if $\ell = \Omega(\lambda)$, verifier accepts with probability $2^{-\Omega(\lambda)}$

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Robust Decomposition

Robustness: If $x \notin \mathcal{L}$, then for all w', at most $2/3$ of $f_i(x', w') = 1$

For false x, no single w' can simultaneously satisfy $f_i(x', \cdot)$; however, all of the $f_i(x', \cdot)$ could individually be satisfiable

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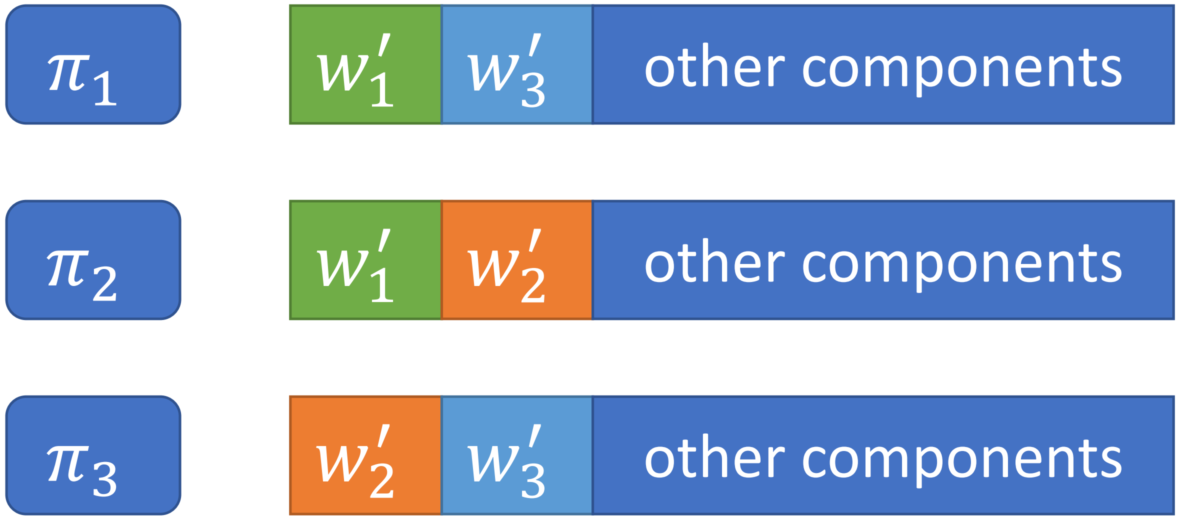
Soundness: Each linear PCP provides $1/\text{poly}(\lambda)$ soundness and for false statement, at least $1/3$ of the statements are false, so if $\ell = \Omega(\lambda)$, verifier accepts with probability $2^{-\Omega(\lambda)}$

Problematic however if prover uses different (x', w') to construct proofs for different f_i's

Consistency Checking

Require that linear PCPs are <u>systematic</u>: linear PCP π contains a copy of the witness:



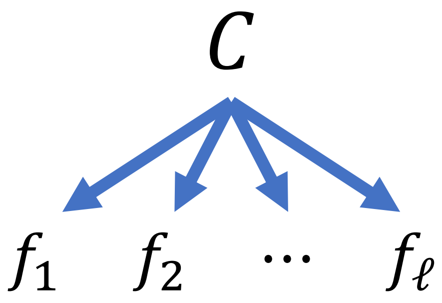
Goal: check that assignments to w' are consistent via linear queries to π_i

First few components of proof correspond to witness associated with the statement → Each proof induces an assignment to a few bits of the common witness w'

[See paper for details]

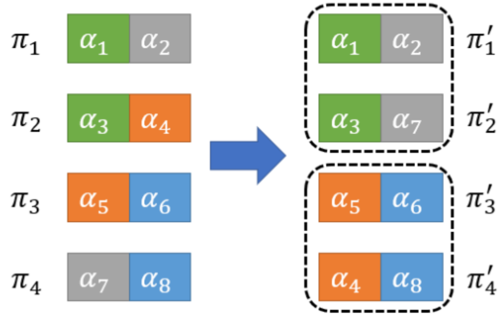
Quasi-Optimal Linear MIPs

Robust Decomposition



- Checking satisfiability of C corresponds to checking satisfiability of $f_1, \ldots, f_\ell$ (each of which can be checked by a circuit of size $|C|/\ell$)
- For a false statement, no single witness can simultaneously satisfy more than a constant fraction of f_i

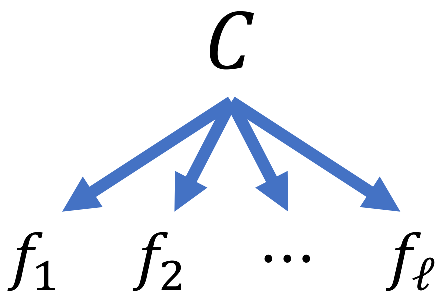
Consistency Check



- Check that consistent witness is used to prove satisfiability of each f_i
- Relies on pairwise consistency checks and permuting the entries to obtain a "nice" replication structure

Quasi-Optimal Linear MIPs

<u>Robust Decomposition</u>



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Robust decomposition can be instantiated by combining "MPC-in-the-head" paradigm [IKOS07] with a robust MPC protocol with polylogarithmic overhead [DIK10]

More generally: viewing a general MPC protocol as a PCP over a large alphabet

[See paper for details]

Conclusions

A SNARG is quasi-optimal if it satisfies the following properties:

- Quasi-optimal succinctness: $|\pi| = \tilde{O}(\lambda)$
- Quasi-optimal prover complexity: $|P| = \tilde{O}(|C|) + \mathrm{poly}(\lambda, \log|C|)$

New framework for building quasi-optimal SNARGs by combining quasi-optimal linear MIP with linear-only vector encryption

- Construction of a quasi-optimal linear MIP possible by combining robust decomposition and consistency check

What if we had a 1-bit SNARG? Implies a form of witness encryption!

[See paper for details]

Open Problems

Quasi-optimal SNARGs with additional properties:

- Publicly-verifiable / multi-theorem (in designated verifier setting)
- Zero-knowledge

Thank you!