Constraining Pseudorandom Functions Privately

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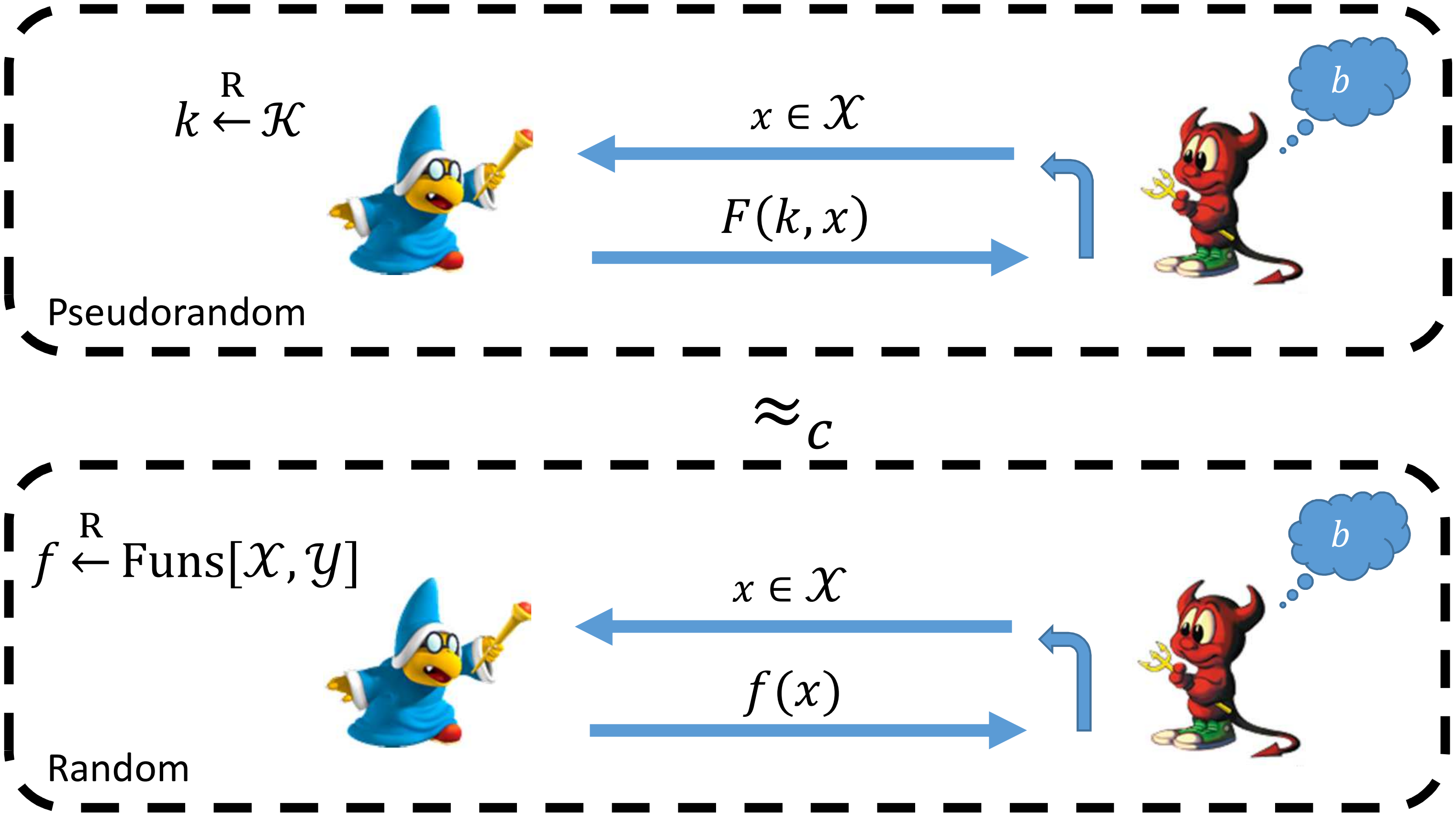
Joint work with Dan Boneh and Kevin Lewi

Pseudorandom Functions (PRFs) [GGM84]



$F: \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$

Constrained PRFs [BW13, BGI13, KPTZ13]

Constrained PRF: PRF with additional "constrain" functionality

PRF key

constrained key

$F: \mathcal{K} \times \mathcal{X} \rightarrow \mathcal{Y}$

can be used to evaluate at all points $x \in \mathcal{X}$ where $C(x) = 1$

Example Constraints

Puncturing:

$$C_z(x) = \begin{cases} 1, & x \neq z \\ 0, & x = z \end{cases}$$

Punctured key can evaluate PRF at all but one point

Example Constraints

Left/right PRF:

- Domain of PRF are tuples (x, y)
- Left constraints:

$$C_z(x,y) = \begin{cases} 1, & x = z \\ 0, & x \neq z \end{cases}$$

Accepts if left components match

- Right constraints:

$$C_z(x,y) = \begin{cases} 1, & y = z \\ 0, & y \neq z \end{cases}$$

Accepts if right components match

- Can be used to build non-interactive identity-based key exchange [BW13]

Constrained PRFs [BW13, BGI13, KPTZ13]

Correctness: constrained evaluation at $x \in \mathcal{X}$ where $C(x) = 1$ yields PRF value at x

Security: PRF value at points $x \in \mathcal{X}$ where $C(x) = 0$ are indistinguishable from random *given* the constrained key

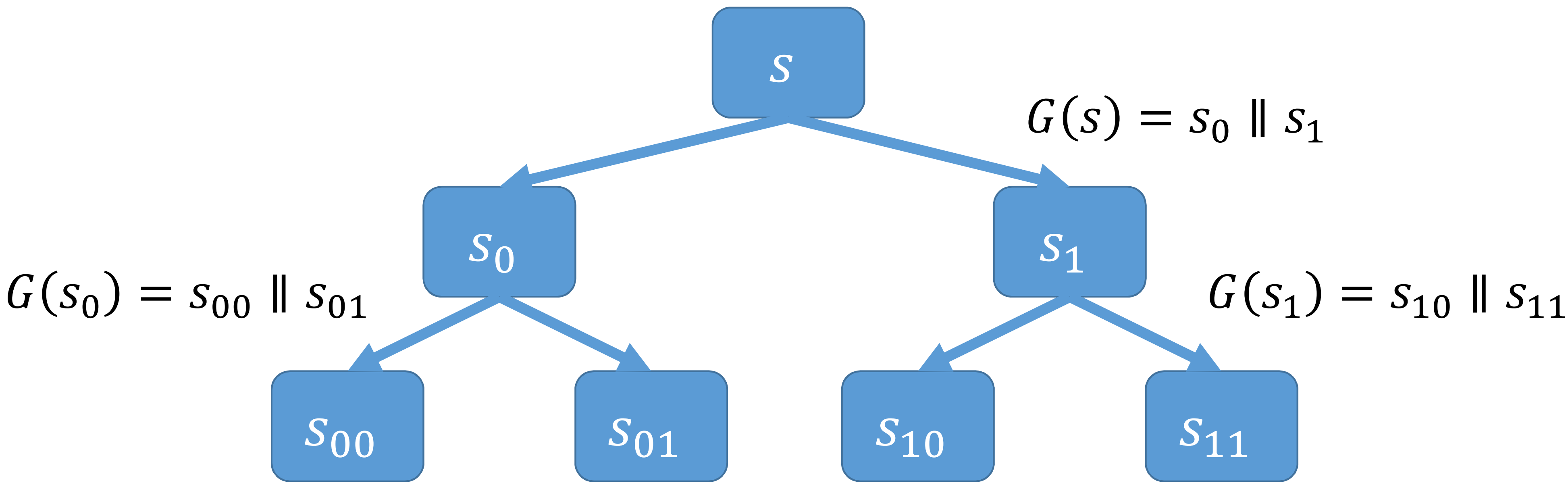
Constrained PRFs [BW13, BGI13, KPTZ13]

Many applications:

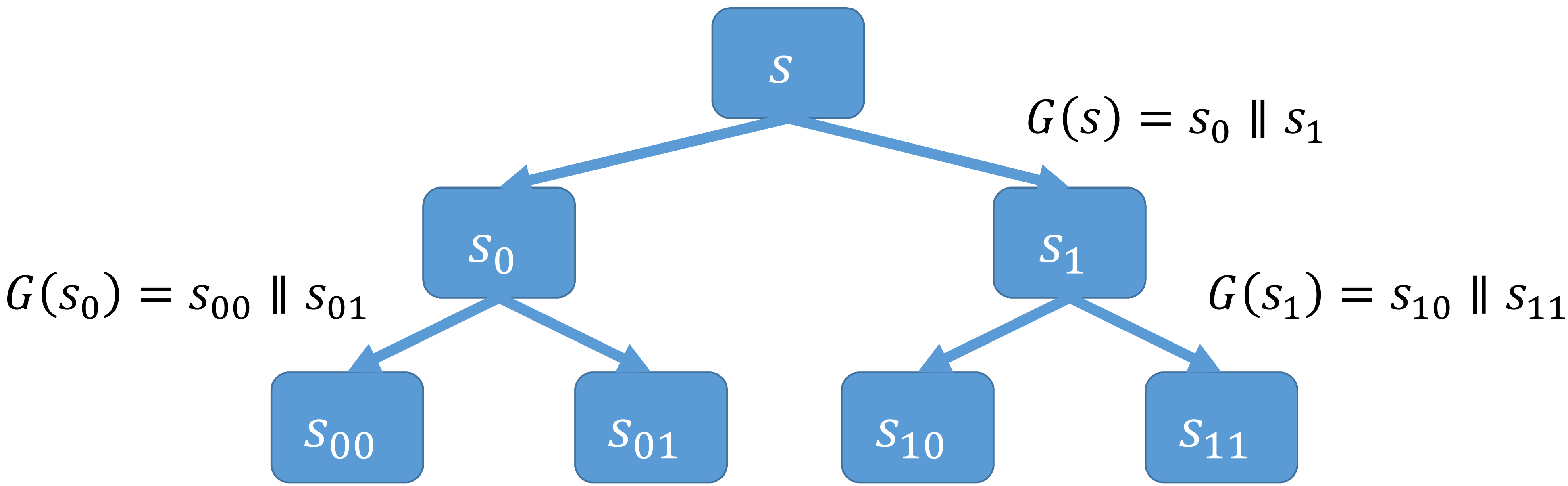
- Identity-Based Key Exchange, Optimal Broadcast Encryption [BW13]
- Punctured Programming Paradigm [SW14]
- Multiparty Key Exchange, Traitor Tracing [BZ14]

Puncturable PRFs from GGM

- Puncturable PRF: constrained keys allow evaluation at *all* but a single point
- Easily constructed from a length-doubling PRG via GGM:

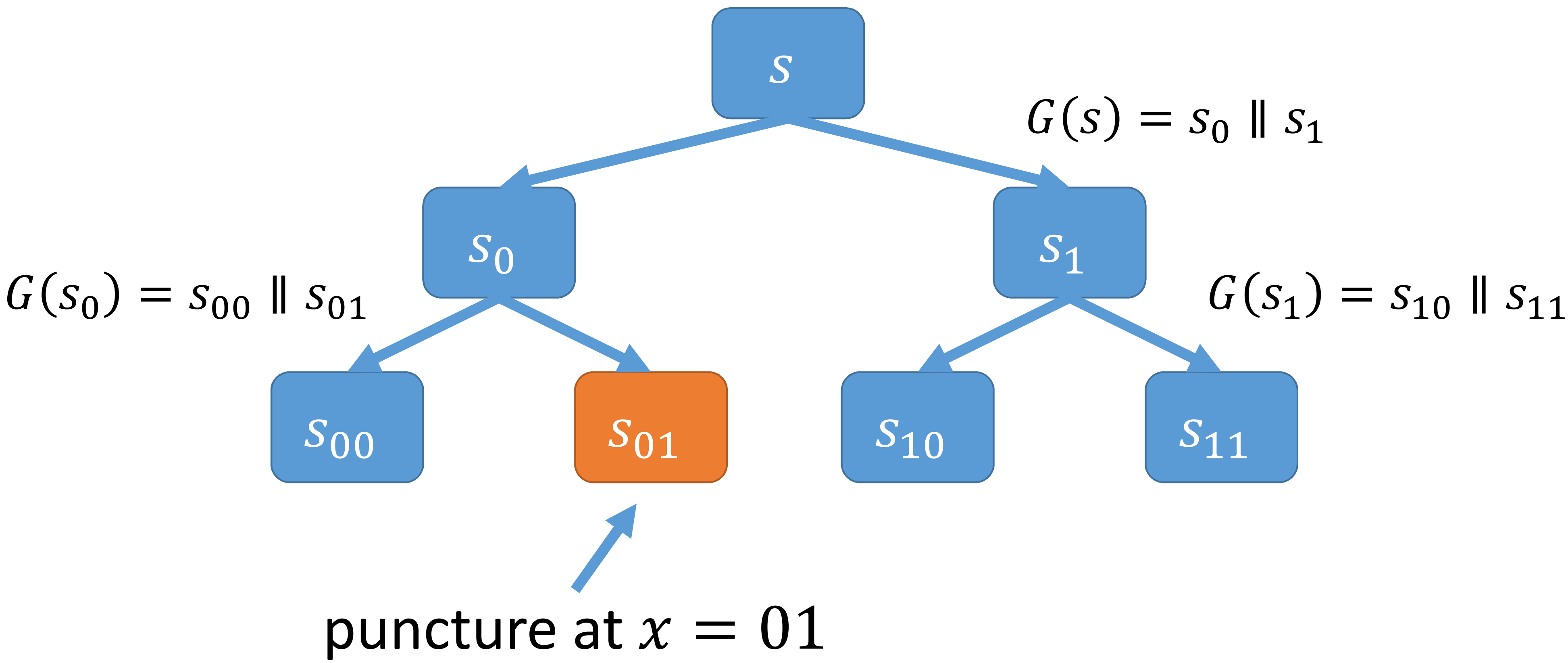


Puncturable PRFs from GGM

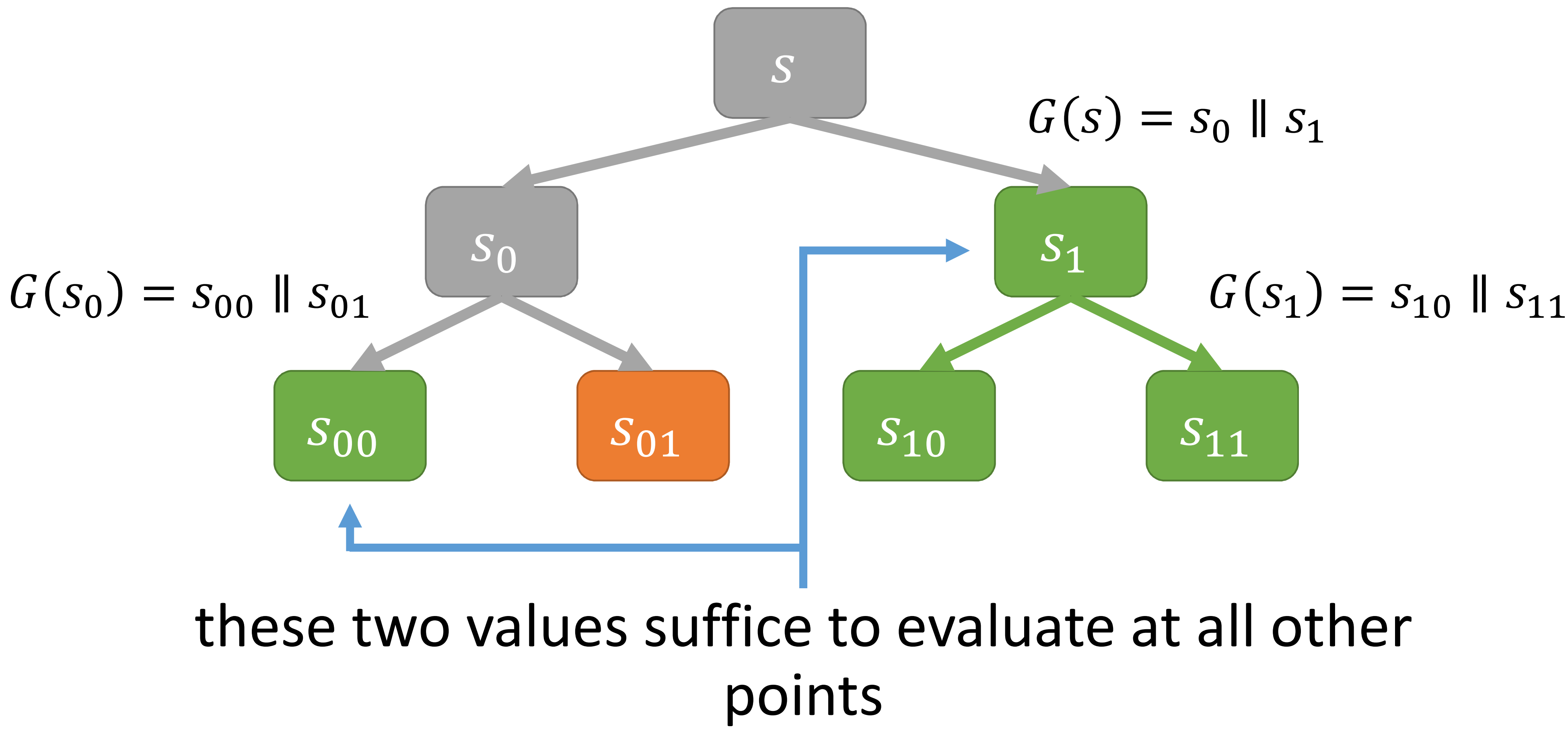


given root key s, can evaluate PRF everywhere

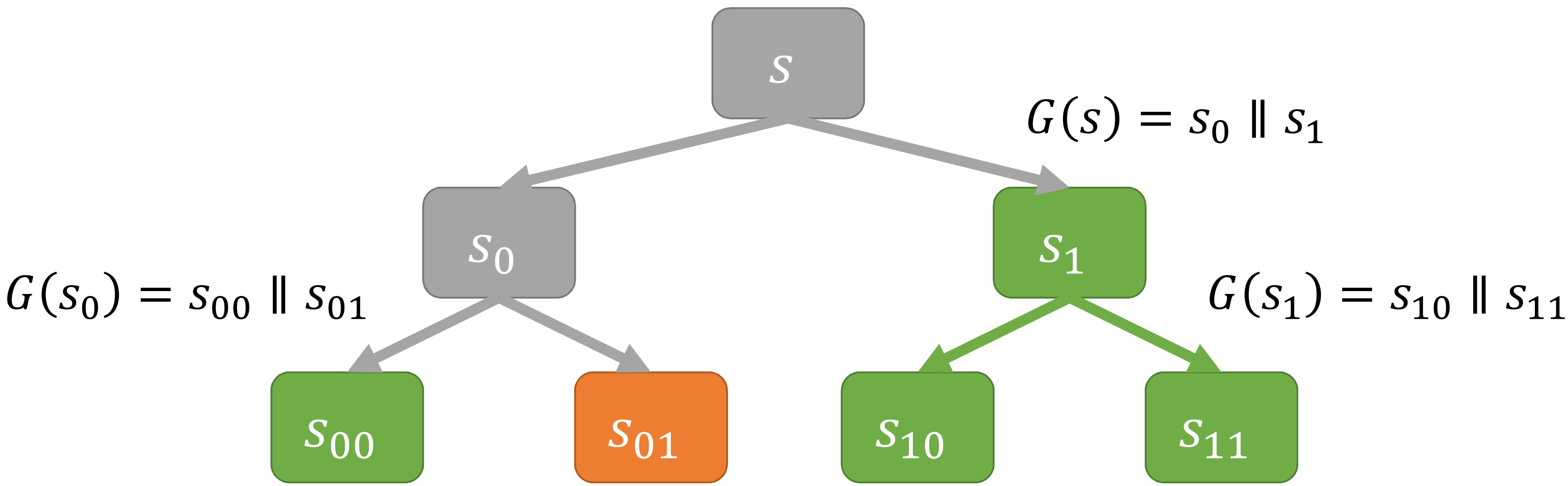
Puncturable PRFs from GGM



Puncturable PRFs from GGM

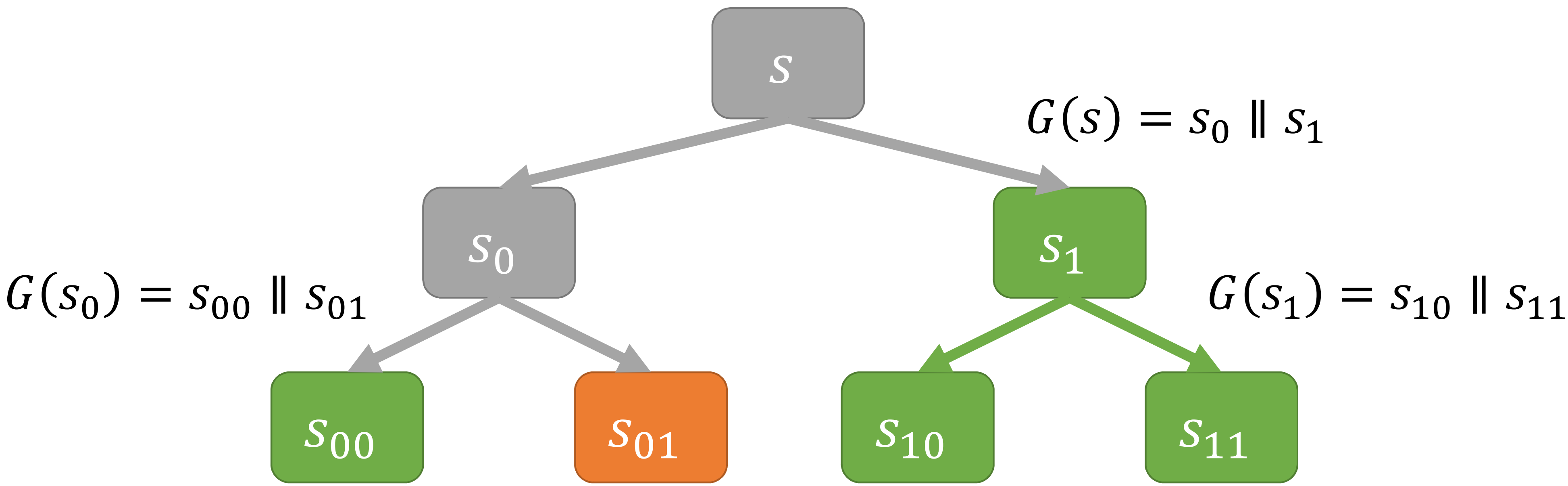


Puncturable PRFs from GGM



in general, punctured key consists of n nodes if domain of PRF is $\{0,1\}^n$

Puncturable PRFs from GGM

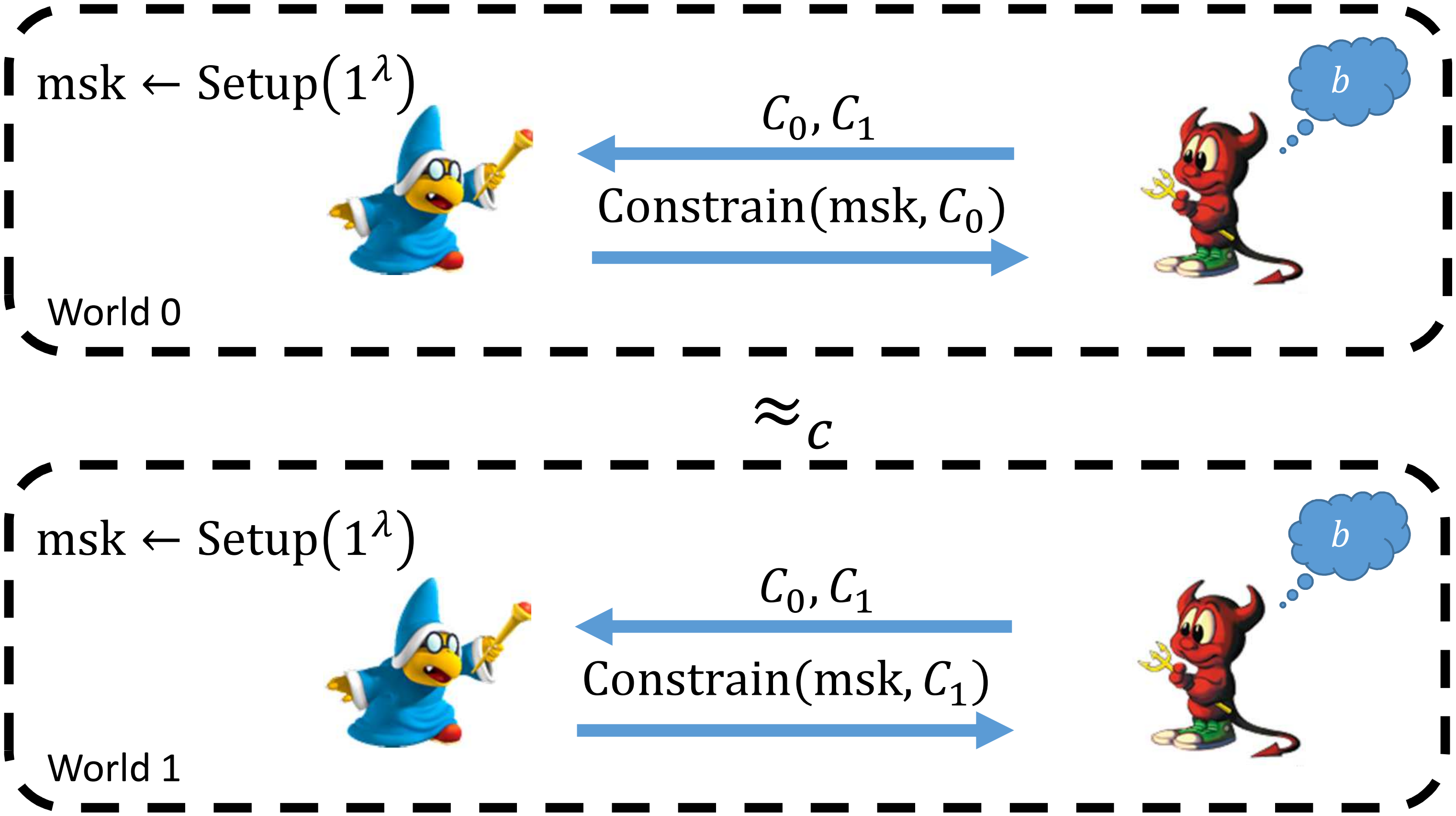


given s_1 and s_{00}, easy to tell that 01 is the punctured point

Constraining PRFs Privately

Can we build a constrained PRF where the constrained key for a circuit C hides C?

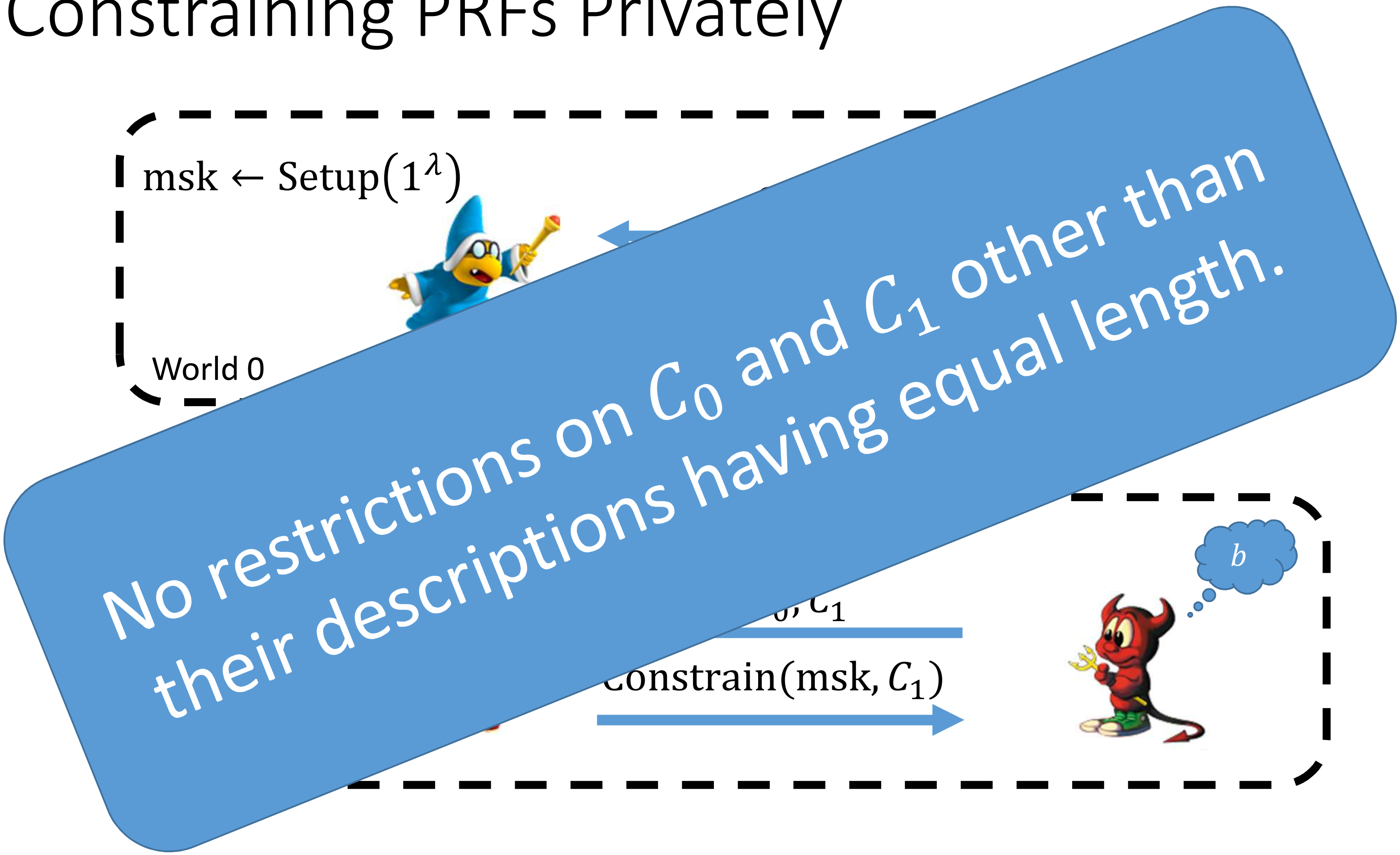
Constraining PRFs Privately



Single-key privacy

Definitions generalize to multi-key privacy. See paper for details.

Constraining PRFs Privately



No restrictions on C_0 and C_1 other than their descriptions having equal length.

Private Puncturing

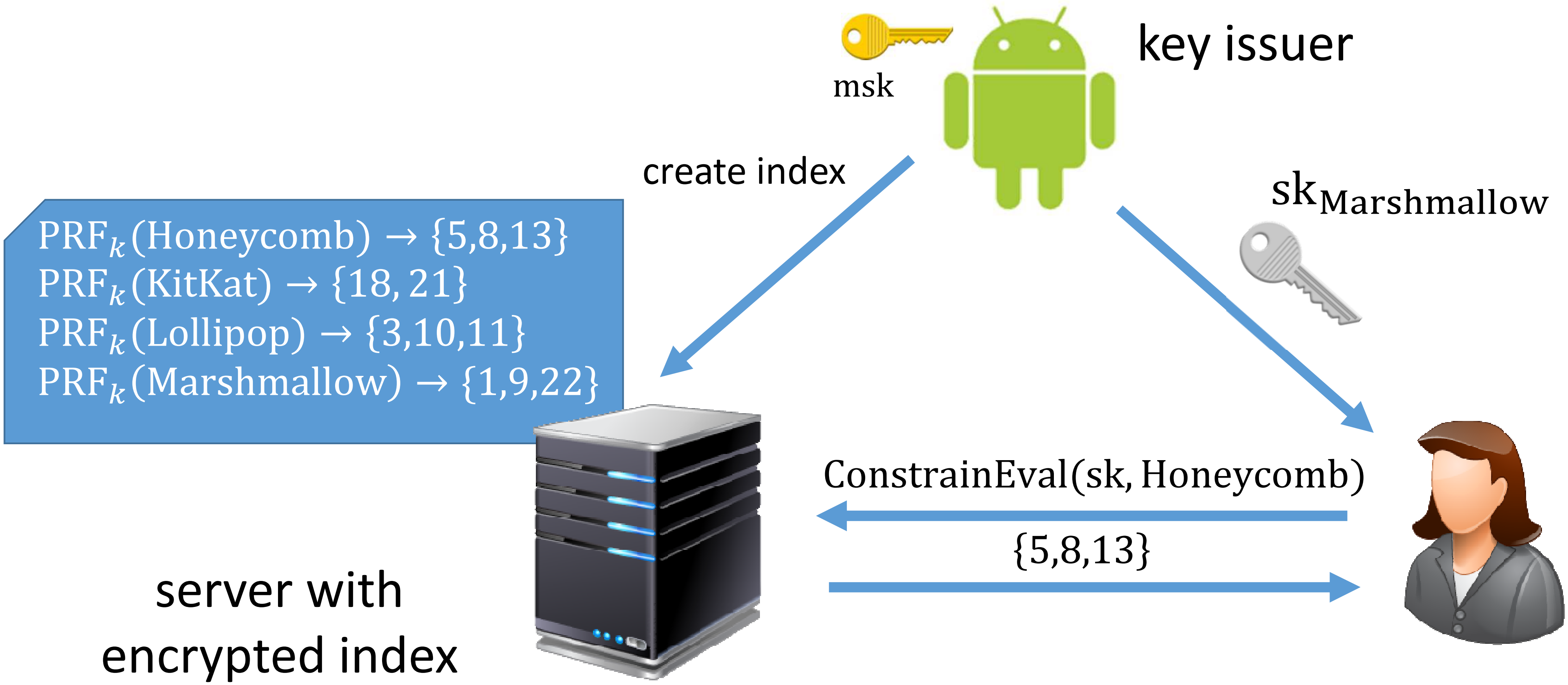
- **Correctness:** constrained evaluation at $x \neq z$ yields $F(k, x)$
- **Security:** $F(k, z)$ is indistinguishable from random
- **Privacy:** constrained key hides z

Implications of Privacy

Consider value of $\mathrm{ConstrainEval}(\mathrm{sk}_z, z)$:

- **Security**: Independent of $F(\mathrm{msk}, z)$
- **Privacy**: Unguessable by the adversary

Using Privacy: Restricted Keyword Search



Using Privacy: Restricted Keyword Search

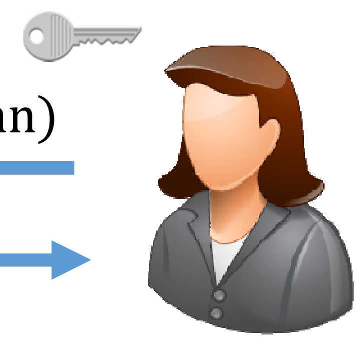
search for non-existent keyword

$\text{PRF}_k(\text{Honeycomb}) \rightarrow \{5,8,13\}$
$\text{PRF}_k(\text{KitKat}) \rightarrow \{18, 21\}$
$\text{PRF}_k(\text{Lollipop}) \rightarrow \{3,10,11\}$
$\text{PRF}_k(\text{Marshmallow}) \rightarrow \{1,9,22\}$

server with encrypted index

ConstrainEval(sk, Jelly Bean)

No results



Using Privacy: Restricted Keyword Search

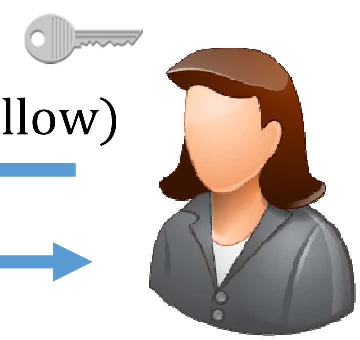
search for “restricted” keyword

$\mathrm{PRF}_k(\mathrm{Honeycomb}) \rightarrow \{5,8,13\}$
$\mathrm{PRF}_k(\mathrm{KitKat}) \rightarrow \{18, 21\}$
$\mathrm{PRF}_k(\mathrm{Lollipop}) \rightarrow \{3,10,11\}$
$\mathrm{PRF}_k(\mathrm{Marshmallow}) \rightarrow \{1,9,22\}$

server with encrypted index

$\mathrm{ConstrainEval}(\mathrm{sk}, \mathrm{Marshmallow})$

No results



Using Privacy: Restricted Keyword Search

- **Security**: $\text{ConstrainEval}(\text{sk}, \text{Marshmallow}) \neq \text{Eval}(\text{msk}, \text{Marshmallow})$
- **Privacy**: Does not learn that no results were returned because no matches for keyword or if the keyword was restricted

$\text{PRF}_k(\text{Honeycomb}) \rightarrow \{5,8,13\}$
$\text{PRF}_k(\text{KitKat}) \rightarrow \{18, 21\}$
$\text{PRF}_k(\text{Lollipop}) \rightarrow \{3,10,11\}$
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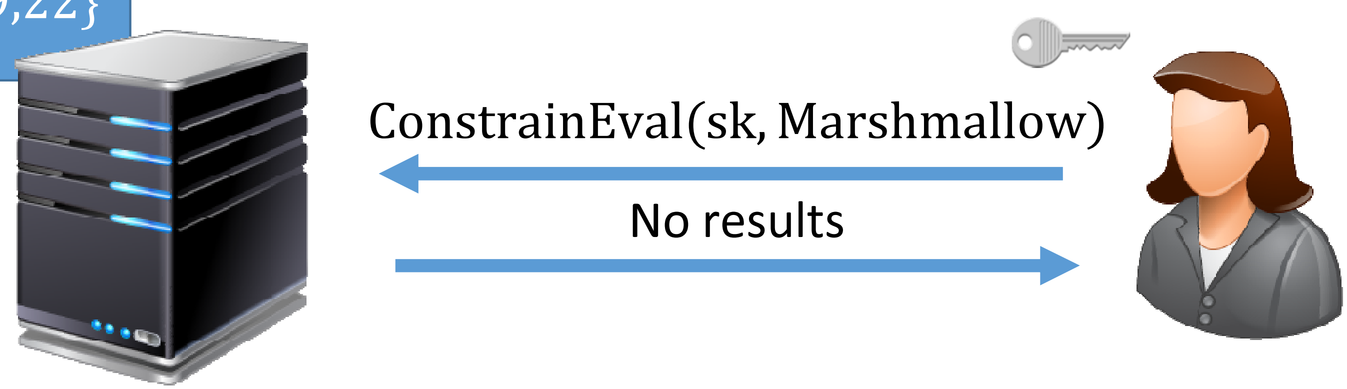


server with encrypted index

The Many Applications of Privacy

- Private constrained MACs
  - Parties can only sign messages satisfying certain policy (e.g., enforce a spending limit), but policies are hidden
- Symmetric Deniable Encryption [CDNO97]
  - Two parties can communicate using a symmetric encryption scheme
  - If an adversary has intercepted a sequence of messages and coerces one of the parties to produce a decryption key for the messages, they can produce a "fake" key that decrypts all but a subset of the messages
- Constructing a family of watermarkable PRFs
  - Can be used to embed a secret message within a PRF that is "unremovable" – useful for authentication [CHNVW15]

See paper for details!

Summary of our Constructions

- From indistinguishability obfuscation (iO):
  - Private puncturable PRFs from iO + one-way functions
  - Private circuit constrained PRFs from sub-exponentially hard iO + one-way functions

- From concrete assumptions on multilinear maps:
  - Private puncturable PRFs from subgroup hiding assumptions

This talk

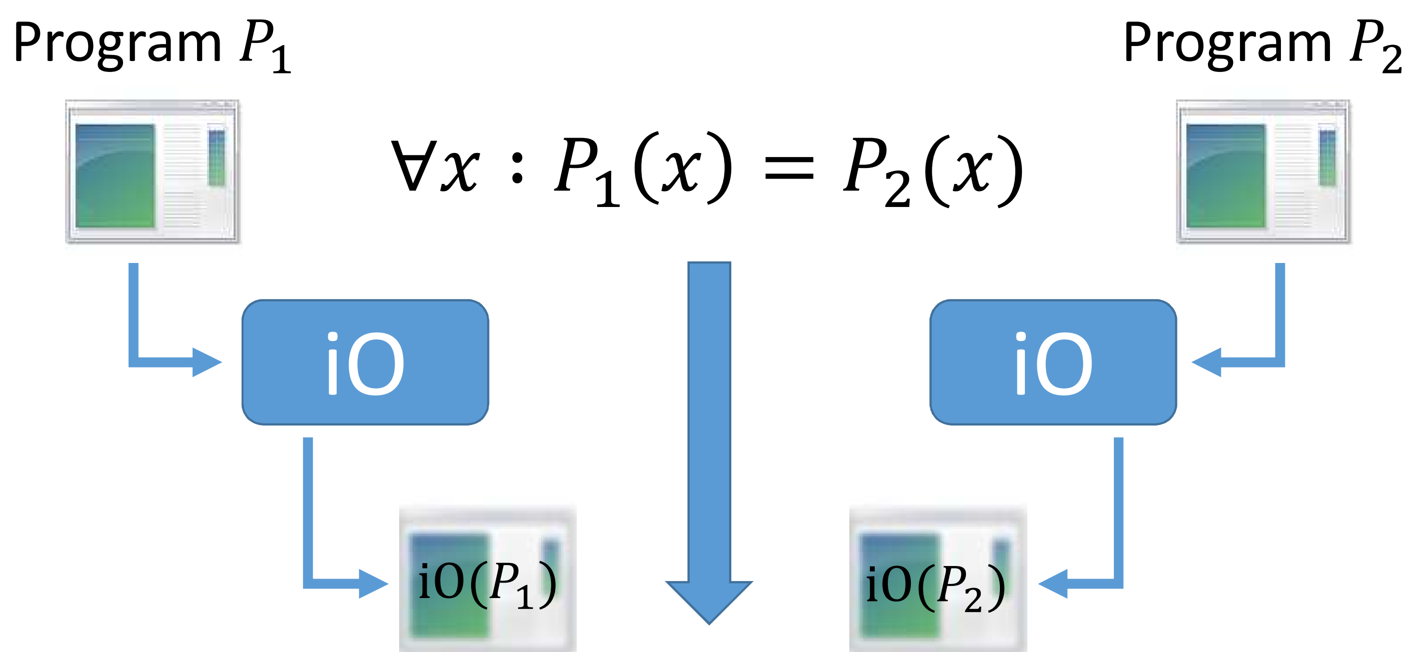
  - Private bit-fixing PRF from multilinear Diffie-Hellman assumption

See paper

Private Puncturing from Indistinguishability Obfuscation

Constructing Private Constrained PRFs

Tool: indistinguishability obfuscation [BGI+01, GGH+13]



$$\mathrm{iO}(P_1) \approx_c \mathrm{iO}(P_2)$$

Indistinguishability Obfuscation (iO)

- First introduced by Barak et al. [BGI$^+$01]
- First construction from multilinear maps [GGH$^+$13]
  - Subsequent constructions from multilinear maps [BR13, BGK$^+$14, AGIS14, Zim14, AB15, …]
  - Constructions also from (compact) functional encryption [AJ15, AJS15]

Indistinguishability Obfuscation (iO)

Many applications – “crypto complete”

- Functional encryption [GGH^{+}13]
- Deniable encryption [SW13]
- Witness encryption [GGSW13]
- Private broadcast encryption [BZ14]
- Traitor tracing [BZ14]
- Multiparty key exchange [BZ14]
- Multiparty computation [GGHR14]
- and more…

Private Puncturing from iO

- Starting point: puncturable PRFs (e.g. GGM)
- Need a way to hide the point that is punctured
  - Intuition: obfuscate the puncturable PRF
- Question: what value to output at the punctured point?

Private Puncturing from iO

Use iO to hide the punctured point and output uniformly random value at punctured point

$P_z(x)$:

- If $x = z$, output r
- Else, output $\mathrm{PRF}(k, x)$

random value (hard coded)

real value of the PRF

Program for punctured PRF (punctured at z)

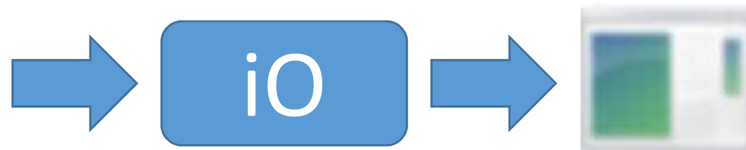
Private Puncturing from iO

Suppose PRF is puncturable (e.g., GGM)

- Master secret key: PRF key k
- PRF output at $x \in \mathcal{X}$: $\mathrm{PRF}(k, x)$

$P_z(x)$:

- If $x = z$, output r
- Else, output $\mathrm{PRF}(k, x)$

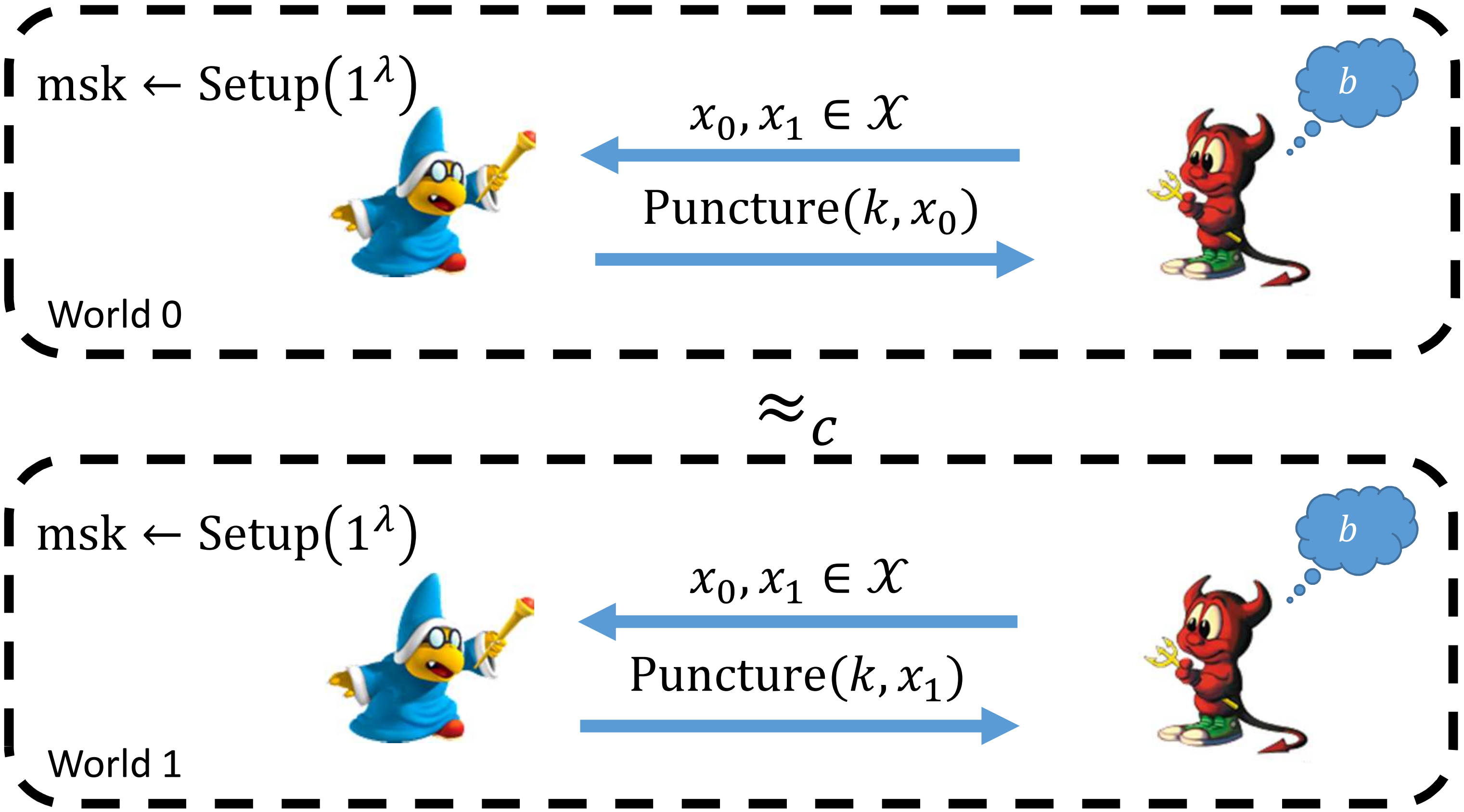


Punctured key for a point z is an obfuscated program

Constrained evaluation corresponds to evaluating obfuscated program

Private Puncturing from iO: Privacy

Recall privacy notion:



Private Puncturing from iO: Privacy

$P_{x_0}(x)$:

- If $x = x_0$, output r
- Else, output $\mathrm{PRF}(k, x)$

Private Puncturing from iO: Privacy

iO ($P_{x_0}(x)$:
- If $x = x_0$, output r
- Else, output $\mathrm{PRF}(k, x)$

) $\approx_c$ iO ($P'_{x_0}(x)$:
- If $x = x_0$, output r
- Else, output $\mathrm{PRF}(k_{x_0}, x)$

)

By correctness of puncturing, P_{x_0} and P'_{x_0} compute identical functions

Private Puncturing from iO: Privacy

iO $\Big($ $P_{x_0}(x)$:

- If $x = x_0$, output r
- Else, output $\mathsf{PRF}(k, x)$

$\Big)$ $\approx_c$ iO $\Big($ $P'_{x_0}(x)$:

- If $x = x_0$, output r
- Else, output $\mathsf{PRF}(k_{x_0}, x)$

$\Big)$

Hybrid 0: Real game

Hybrid 1: Challenger responds to puncture query with iO of this program

Private Puncturing from iO: Privacy

Invoke puncturing security

Given punctured key k_{x_0}, cannot distinguish real value $\mathrm{PRF}(k, x_0)$ from uniformly random value

iO ($P'_{x_0}(x)$:

- If $x = x_0$, output r
- Else, output $\mathrm{PRF}(k_{x_0}, x)$)

Hybrid 1

Private Puncturing from iO: Privacy

Invoke puncturing security

Given punctured key k_{x_0}, cannot distinguish real value $\mathrm{PRF}(k, x_0)$ from uniformly random value

iO ($P'_{x_0}(x)$:
- If $x = x_0$, output r
- Else, output PRF(k_{x_0}, x)

) $\approx_c$ iO ($P''_{x_0}(x)$:
- If $x = x_0$, output PRF(k, x_0)
- Else, output PRF(k_{x_0}, x)

)

Hybrid 1

Hybrid 2

Private Puncturing from iO: Privacy

iO $\left(\begin{array}{l} P''_{x_0}(x): \\ \bullet \text{ If } x = x_0\text{, output PRF}(k, x_0) \\ \bullet \text{ Else, output PRF}(k_{x_0}, x) \end{array} \right)$

Hybrid 2

Private Puncturing from iO: Privacy

Invoke iO security

iO ($P''_{x_0}(x)$:
- If $x = x_0$, output $\text{PRF}(k, x_0)$
- Else, output $\text{PRF}(k_{x_0}, x)$)

$\approx_c$ iO ($P'''_{x_0}(x)$:
- Output $\text{PRF}(k, x)$)

Hybrid 2 Hybrid 3

The program in Hybrid 3 is independent of x_0. Similar argument holds starting from $P_{x_1}(x)$

Private Puncturing from iO: Summary

Use iO to hide the punctured point and output uniformly random value at punctured point

$P_z(x)$:

- If $x = z$, output r
- Else, output $\mathrm{PRF}(k, x)$

Private Circuit Constrained PRF from iO

Construction generalizes to circuit constraints, except random values now derived from another PRF

$P_C(x)$:

- If $C(x) = 0$, output $\mathrm{PRF}(k', x)$
- If $C(x) = 1$, output $\mathrm{PRF}(k, x)$

k' is independently sampled PRF key

"real" PRF value

Private Circuit Constrained PRF from iO

$P_C(x)$:

- If $C(x) = 0$, output $\text{PRF}(k', x)$
- If $C(x) = 1$, output $\text{PRF}(k, x)$

Recall intuitive requirements for private constrained PRF:

- **Security**: Values at constrained points independent of actual PRF value at those points
- **Privacy**: Values at constrained points are unguessable by the adversary

Private Circuit Constrained PRF from iO

$P_C(x)$:

- If $C(x) = 0$, output PRF(k', x)
- If $C(x) = 1$, output PRF(k, x)

Security proof similar to that for private puncturable PRF

Number of hybrids equal to number of points that differ across the two circuits, so sub-exponential hardness needed in general

Private Puncturing from Multilinear Maps

Private Puncturing from Multilinear Maps

- Composite-order (ideal) multilinear maps* [BS04]
  - Fix <u>composite</u> modulus $N = pq$
  - Base group $\mathbb{G}_1$ and target group $\mathbb{G}_n$ (of order N) with canonical generators g_1 and g_n, respectively
  - Multilinear map operation:
    $$e\left(g_1^{\alpha_1}, g_1^{\alpha_2}, \ldots, g_1^{\alpha_n}\right) = g_n^{\alpha_1 \alpha_2 \cdots \alpha_n}$$

*For simplicity, we describe our construction using ideal multilinear maps. It is straightforward to translate our construction to use composite-order graded multilinear encodings [CLT13]

Private Puncturing from Multilinear Maps

- Composite-order (ideal) multilinear maps [BS04]
  - Let $\mathbb{G}_{1,p}$ be subgroup of order p of $\mathbb{G}_1$
  - Subgroup decision assumption [BGN05]: hard to distinguish random elements of the full group $\mathbb{G}_1$ from random elements of the subgroup $\mathbb{G}_{1,p}$

Private Puncturing from Multilinear Maps

Starting point: multilinear analog of Naor-Reingold [NR97, BW13]

master secret key:

$g_1^{\alpha_{1,0}}$	$g_1^{\alpha_{2,0}}$	$\cdots$	$g_1^{\alpha_{n,0}}$
$g_1^{\alpha_{1,1}}$	$g_1^{\alpha_{2,1}}$	$\cdots$	$g_1^{\alpha_{n,1}}$

collection of $2n$ random group elements from $\mathbb{G}_1$

Private Puncturing from Multilinear Maps

PRF evaluation via multilinear map

$g_1^{\alpha_{1,0}}$	$g_1^{\alpha_{2,0}}$	$g_1^{\alpha_{3,0}}$	$g_1^{\alpha_{4,0}}$	$g_1^{\alpha_{5,0}}$
$g_1^{\alpha_{1,1}}$	$g_1^{\alpha_{2,1}}$	$g_1^{\alpha_{3,1}}$	$g_1^{\alpha_{4,1}}$	$g_1^{\alpha_{5,1}}$

Private Puncturing from Multilinear Maps

PRF evaluation via multilinear map

$g_1^{\alpha_{1,0}}$	$g_1^{\alpha_{2,0}}$	$g_1^{\alpha_{3,0}}$	$g_1^{\alpha_{4,0}}$	$g_1^{\alpha_{5,0}}$
$g_1^{\alpha_{1,1}}$	$g_1^{\alpha_{2,1}}$	$g_1^{\alpha_{3,1}}$	$g_1^{\alpha_{4,1}}$	$g_1^{\alpha_{5,1}}$

$$F_k(01101) = e\left(g_1^{\alpha_{1,0}}, g_1^{\alpha_{2,1}}, g_1^{\alpha_{3,1}}, g_1^{\alpha_{4,0}}, g_1^{\alpha_{5,1}}\right)$$

Private Puncturing from Multilinear Maps

Puncture PRF by exploiting orthogonality

master secret key:

$g_{1,p}^{\alpha_{1,0}}$	$g_{1,p}^{\alpha_{2,0}}$	$g_{1,p}^{\alpha_{3,0}}$	$g_{1,p}^{\alpha_{4,0}}$	$g_{1,p}^{\alpha_{5,0}}$
$g_{1,p}^{\alpha_{1,1}}$	$g_{1,p}^{\alpha_{2,1}}$	$g_{1,p}^{\alpha_{3,1}}$	$g_{1,p}^{\alpha_{4,1}}$	$g_{1,p}^{\alpha_{5,1}}$

all elements in subgroup

puncture at 01101:

$g_{1}^{\alpha_{1,0}}$	$g_{1,p}^{\alpha_{2,0}}$	$g_{1,p}^{\alpha_{3,0}}$	$g_{1}^{\alpha_{4,0}}$	$g_{1,p}^{\alpha_{5,0}}$
$g_{1,p}^{\alpha_{1,1}}$	$g_{1}^{\alpha_{2,1}}$	$g_{1}^{\alpha_{3,1}}$	$g_{1,p}^{\alpha_{4,1}}$	$g_{1}^{\alpha_{5,1}}$

punctured components in full group

Private Puncturing from Multilinear Maps

Correctness

puncture at $x^* = 01101$:

$g_1^{\alpha_{1,0}}$	$g_{1,p}^{\alpha_{2,0}}$	$g_{1,p}^{\alpha_{3,0}}$	$g_1^{\alpha_{4,0}}$	$g_{1,p}^{\alpha_{5,0}}$
$g_{1,p}^{\alpha_{1,1}}$	$g_1^{\alpha_{2,1}}$	$g_1^{\alpha_{3,1}}$	$g_{1,p}^{\alpha_{4,1}}$	$g_1^{\alpha_{5,1}}$

Correctness by multilinearity (and CRT):

$$e\left(g_1^{\beta_1}, \dots, g_1^{\beta_n}\right) = e\left(g_{1,p}, \dots, g_{1,p}\right)^{\beta_1 \cdots \beta_n \pmod p} e\left(g_{1,q}, \dots, g_{1,q}\right)^{\beta_1 \cdots \beta_n \pmod q}$$

For all $x \neq x^*$, there is some i where $x_i \neq x_i^*$ so $\beta_{i,x_i^*} = 0 \pmod q$ where $\left(g^{\beta_{i,0}}, g^{\beta_{i,1}}\right)$ is the i^{th} component of the secret key

Private Puncturing from Multilinear Maps

Privacy

puncture at $x^* = 01101$:

$g_1^{\alpha_{1,0}}$	$g_{1,p}^{\alpha_{2,0}}$	$g_{1,p}^{\alpha_{3,0}}$	$g_1^{\alpha_{4,0}}$	$g_{1,p}^{\alpha_{5,0}}$
$g_{1,p}^{\alpha_{1,1}}$	$g_1^{\alpha_{2,1}}$	$g_1^{\alpha_{3,1}}$	$g_{1,p}^{\alpha_{4,1}}$	$g_1^{\alpha_{5,1}}$

Follows directly by subgroup decision: elements of $\mathbb{G}_1$ look indistinguishable from elements of $\mathbb{G}_{1,p}$

Private Puncturing from Multilinear Maps

Puncturing Security

puncture at
$x^* = 01101$:

$g_1^{\alpha_{1,0}}$	$g_{1,p}^{\alpha_{2,0}}$	$g_{1,p}^{\alpha_{3,0}}$	$g_1^{\alpha_{4,0}}$	$g_{1,p}^{\alpha_{5,0}}$
$g_{1,p}^{\alpha_{1,1}}$	$g_1^{\alpha_{2,1}}$	$g_1^{\alpha_{3,1}}$	$g_{1,p}^{\alpha_{4,1}}$	$g_1^{\alpha_{5,1}}$

Follows from a multilinear Diffie-Hellman subgroup decision assumption on composite-order multilinear maps

See paper for details!

Conclusions

- New notion of <u>private</u> constrained PRFs
- Simple definitions, but require powerful tools to construct: iO / multilinear maps
- Private constrained PRFs immediately provide natural solutions to many problems

Open Questions

- Puncturable PRFs can be constructed from OWFs
  - Can we construct private puncturable PRFs from OWFs?
  - Does private puncturing necessitate strong assumptions like multilinear maps?
  - Can we construct private circuit-constrained PRFs without requiring sub-exponentially hard iO?
- Most of our candidate applications just require private puncturable PRFs
  - New applications for more expressive families of constraints?

Questions?

https://eprint.iacr.org/2015/1167.pdf