A Somewhat Informal Introduction to FHE

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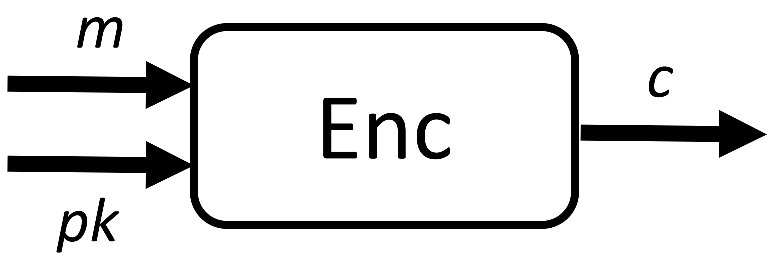
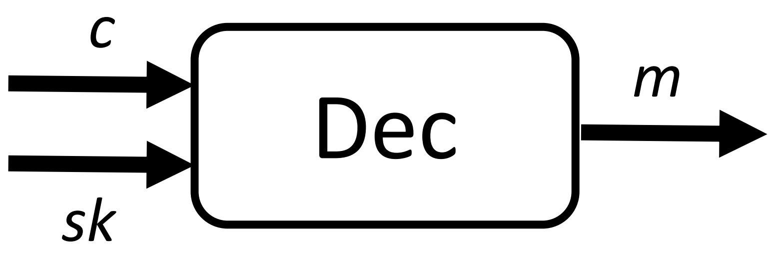
Basic Definitions

Homomorphic Encryption

Homomorphic encryption scheme: encryption scheme that allows computation on ciphertexts

Comprises of three functions:



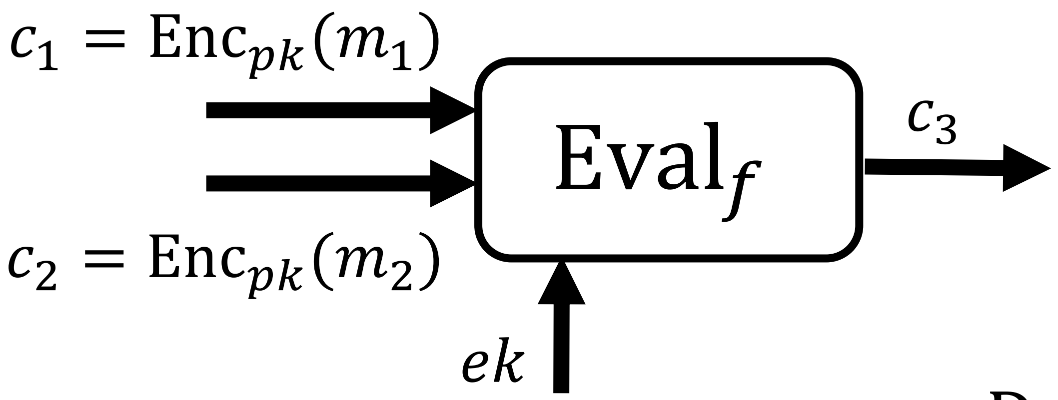


Must satisfy usual notion of semantic security

Homomorphic Encryption

Homomorphic encryption scheme: encryption scheme that allows computation on ciphertexts

Comprises of three functions:



$$\mathrm{Dec}_{sk}\left(\mathrm{Eval}_f(ek, c_1, c_2)\right) = f(m_1, m_2)$$

Fully Homomorphic Encryption (FHE)

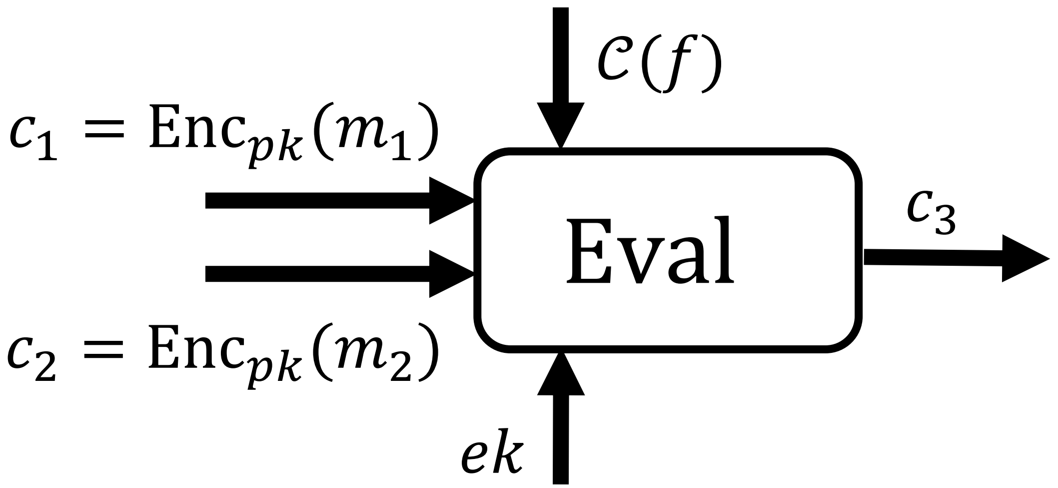
Many homomorphic encryption schemes:

- ElGamal: $f(m_0, m_1) = m_0 m_1$
- Paillier: $f(m_0, m_1) = m_0 + m_1$
- Goldwasser-Micali: $f(m_0, m_1) = m_0 \oplus m_1$

Fully homomorphic encryption: homomorphic with respect to **two** operations: addition and multiplication

- Can evaluate Boolean and arithmetic circuits
- [BGN05]: one multiplication, many additions
- [Gen09]: first FHE construction from lattices

Fully Homomorphic Encryption



$\mathcal{C}(f)$: circuit for some function f

Correctness: $\mathrm{Dec}_{sk}\left(\mathrm{Eval}_f(ek, c_1, c_2)\right) = f(m_1, m_2)$

Circuit Privacy: $\mathrm{Enc}_{pk}\big(\mathcal{C}(m_1, m_2)\big) \approx \mathrm{Eval}_f(ek, c_1, c_2)$

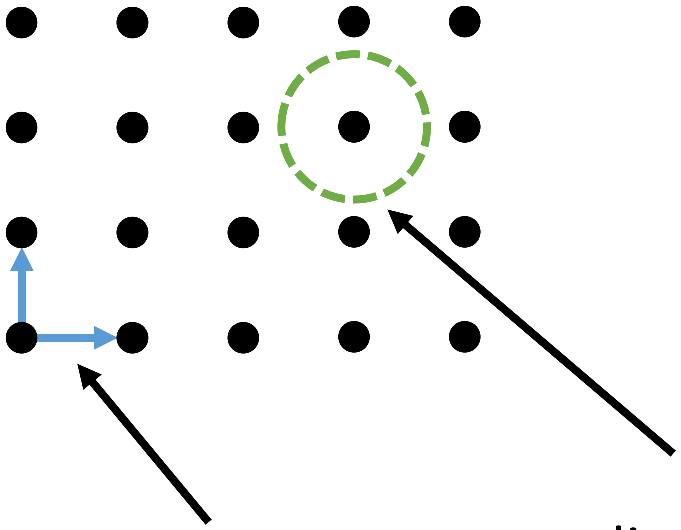
Compactness: Decryption circuit has size at most $\mathrm{poly}(\lambda)$

Lattices and LWE

Lattices

All known FHE constructions based on lattice problems

Lattices are discrete additive subgroups



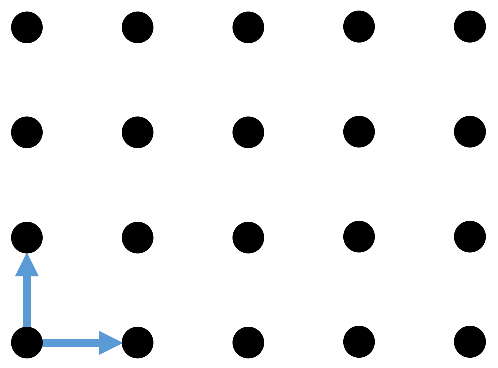
equivalent definition: the set of **integer** combination of basis vectors

basis vectors

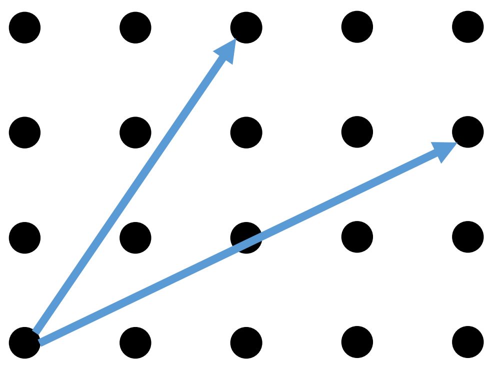
discrete subgroup: no other lattice point contained in ball of radius $\epsilon > 0$ around each lattice point

Hard Lattice Problems

Finding a short vector in a lattice (SVP)



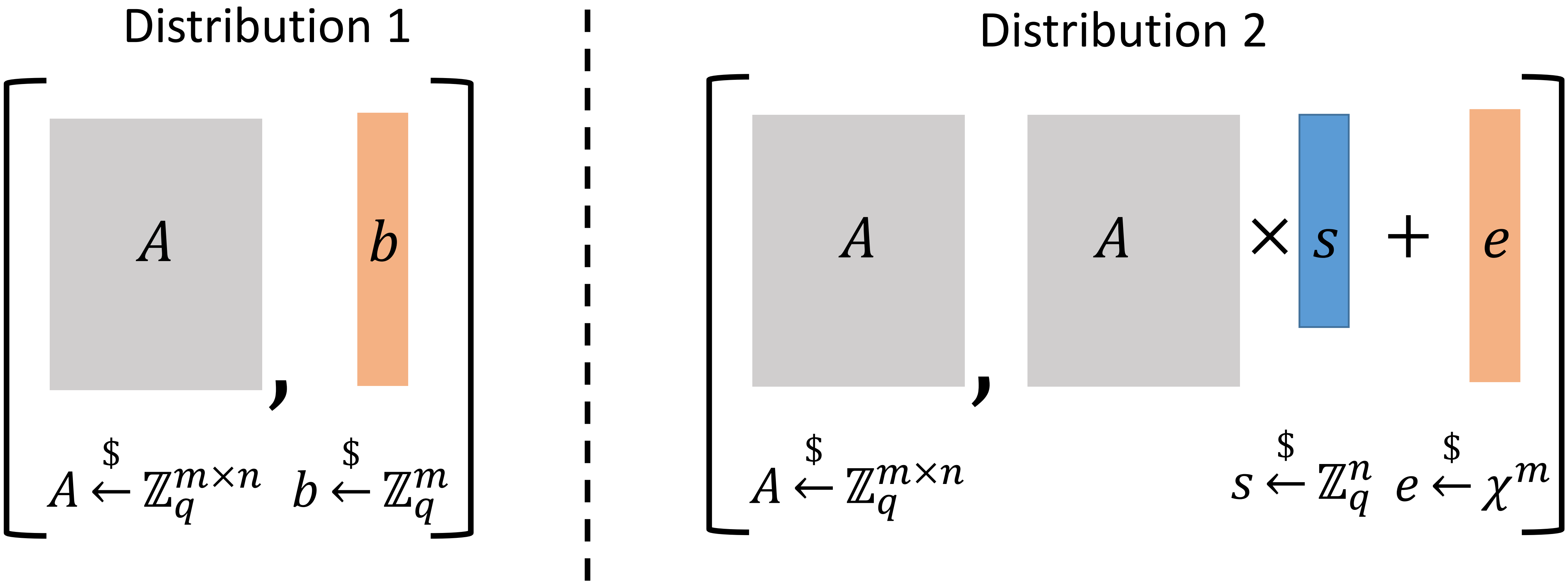
“Good” basis: easy



“Bad” basis: not so easy

Exact SVP is NP-hard. Approximation algorithms try to find a “good” basis using lattice-reduction techniques

Learning with Errors (LWE) [Reg05]



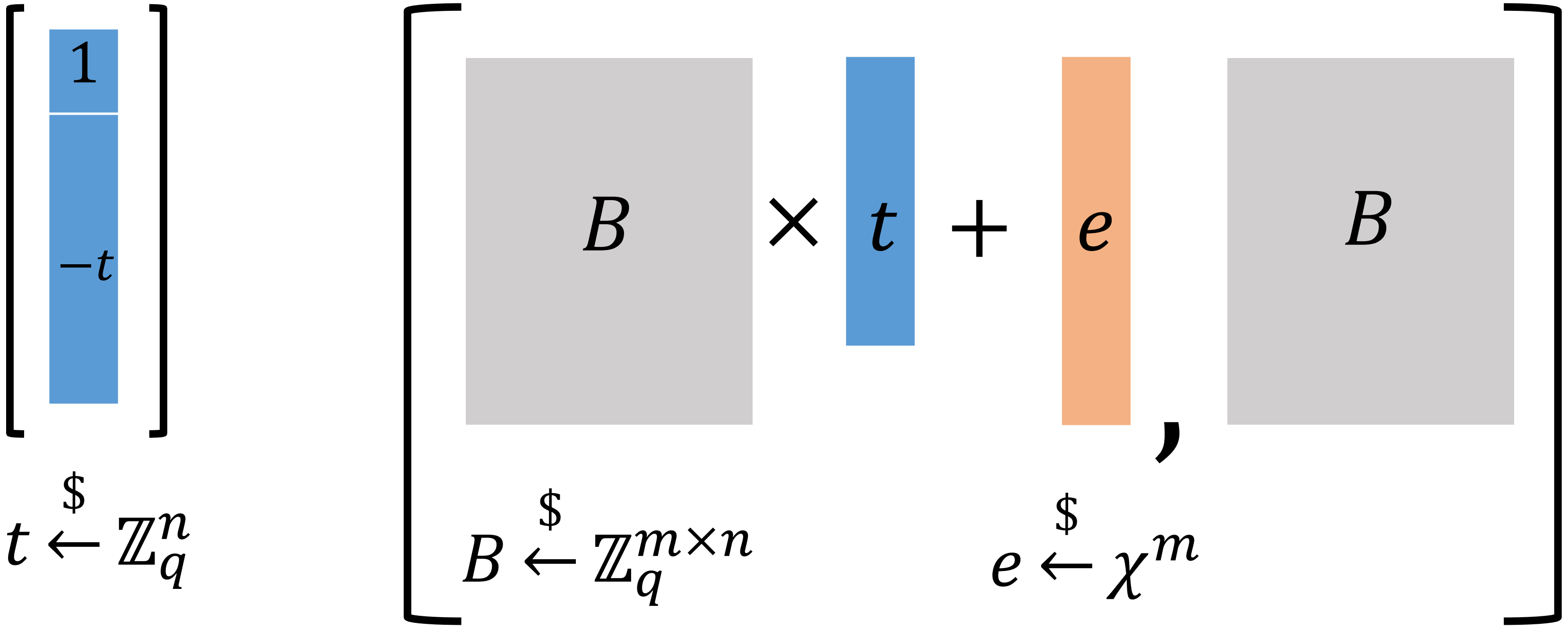
LWE Assumption: distributions 1 and 2 are computationally indistinguishable

Learning with Errors (LWE)

A gold mine of applications!

- PKC: [Reg05], [KTX07], [Pei09]
- FHE: [BV11], [BGV12], [Bra12], [GSW13]
- IBE: [GPV08], [CHKP10], [ABB10]
- ABE: [GVW13], [BCG+14]
- FE: [AFV11]
- … and many more!

Public Key Encryption from LWE [Reg05]



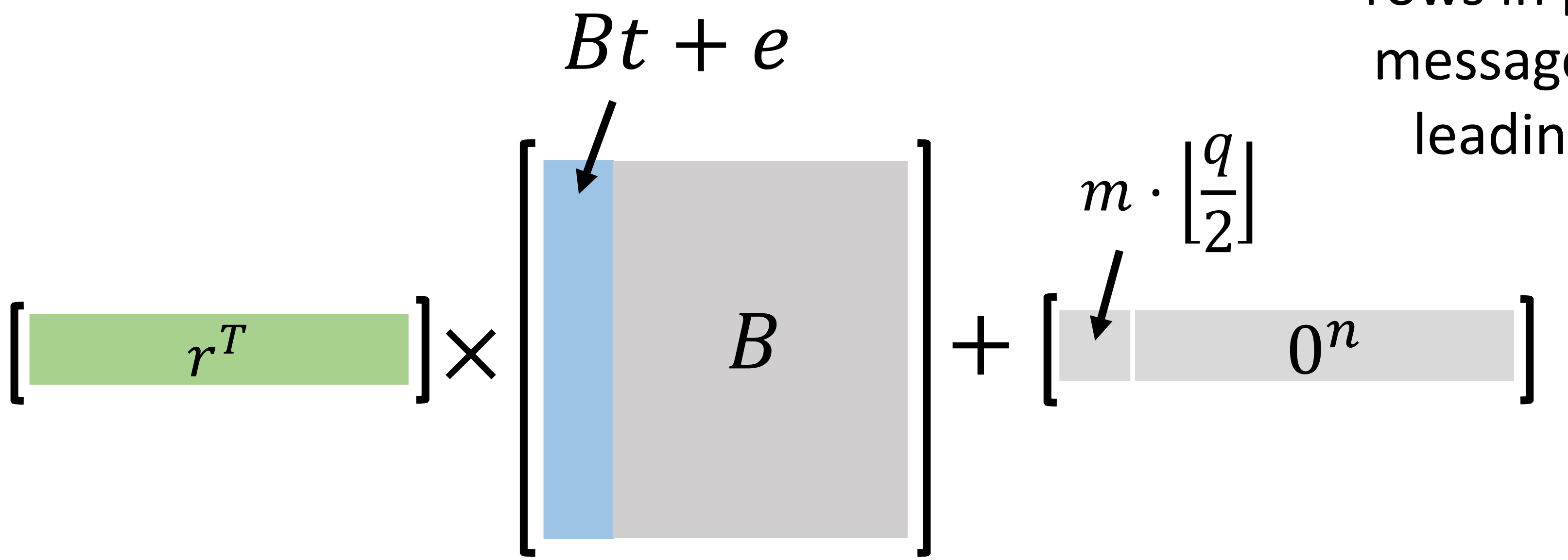
secret key s

public key A

secret key is LWE secret, public key consists of LWE samples

Regev Encryption

random subset sum of rows in public key, with message embedded in leading component

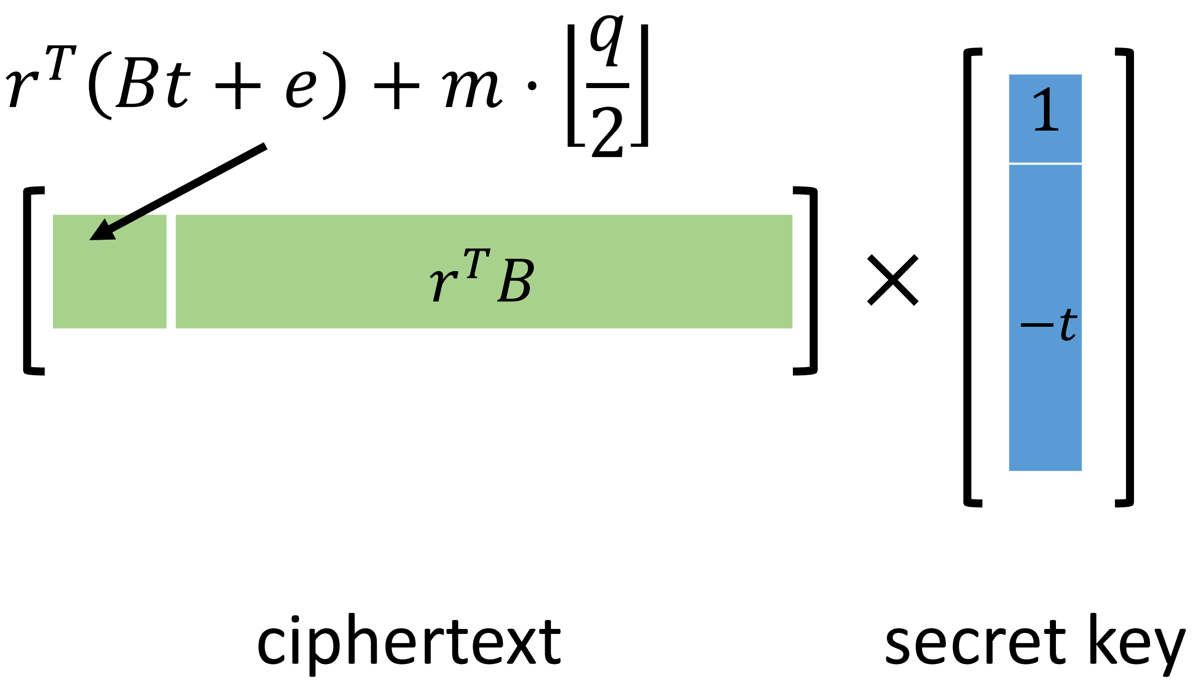


$r \xleftarrow{\$} \{0,1\}^m$

public key

$m \in \{0,1\}$

Regev Decryption



$$= r^T Bt + r^T e + m \cdot \left\lfloor \frac{q}{2} \right\rfloor - r^T Bt$$

$$= m \cdot \left\lfloor \frac{q}{2} \right\rfloor + r^T e$$

multiplying by $\frac{2}{q}$ recovers the message if $r^T e$ is small

PKC from LWE: Regev Encryption [Reg05]

- **Private key:** choose $t \xleftarrow{\$} \mathbb{Z}_q^n$ and set $s \leftarrow (1, -t)$
- **Public key:** Choose $B \xleftarrow{\$} \mathbb{Z}_q^{m \times n}$, $e \xleftarrow{\$} \chi^m$ and compute

$$A \leftarrow (Bs + e, B) \in \mathbb{Z}_q^{m \times (n+1)}$$

- **Encrypt:** Choose random 0/1 vector $r \xleftarrow{\$} \{0,1\}^m$ and compute

$$r^T A + \left(m \cdot \left\lfloor \frac{q}{2} \right\rfloor, 0^n\right) \in \mathbb{Z}_q^{n+1}$$

- **Decrypt:** To decrypt ciphertext c, compute $\left\lfloor \frac{2}{q} \langle c, s \rangle \right\rceil$

PKC from LWE: Regev Encryption [Reg05]

Correctness: if error sufficiently small $\left(< \frac{q}{4}\right)$, then rounding yields the underlying message.

Security: random subset sum of (a_i, b_i) is statistically close to uniform (argument based on leftover hash lemma). Security follows by LWE assumption.

PKC from LWE: Regev Encryption [Reg05]

Key intuition: hide message by adding some noise; everything works if noise is sufficiently small

Basic observation underlying many FHE constructions

SWHE Construction from LWE

From SWHE to FHE

- Somewhat homomorphic encryption: encryption scheme that supports a *limited* number of operations
- All known constructions based on lattices:
  - Hide messages by adding noise
  - Homomorphic operations increase noise
- Gentry's blueprint [Gen09]: bootstrapping SWHE to FHE
  - Homomorphically evaluate the decryption circuit
  - Provides a way to "refresh" a ciphertext

A Simple SWHE Scheme [GSW13]

- Ciphertext are matrices
- Secret key is a vector $v \in \mathbb{Z}_q^n$
- A ciphertext C encrypts a message m if the following holds:

  $$Cv = mv + e$$

  where e is a small error term
- **Intuition:** the message is an *approximate* eigenvalue of the ciphertext

The GSW Scheme

- A ciphertext C encrypts a message m if the following holds:

$$Cv = mv + e$$

  where e is a small error term
- Can decrypt if v has a "big" coefficient v_i by rounding:

$$\left\lfloor \frac{\langle C_i, v \rangle}{v_i} \right\rceil = \left\lfloor \frac{mv_i + e}{v_i} \right\rceil$$

  where C_i denotes the i^{th} row of C

The GSW Scheme

- Homomorphic operations very natural – suppose C_1 encrypts m_1 and C_2 encrypts m_2

- Homomorphic addition: $C_1 + C_2$ (almost) encrypts $m_1 + m_2$:

$$(C_1 + C_2)v = (m_1 + m_2)v + \boxed{e_1 + e_2}$$

- Homomorphic multiplication: $C_1 C_2$ (almost) encrypts $m_1 m_2$:

$$C_1 C_2 v = (m_1 m_2)v + \boxed{m_2 e_1 + C_1 e_2}$$

- Everything works if noise is small enough

Constraining Noise Growth

- Recall Regev decryption:

$$m \leftarrow \left\lfloor \frac{2}{q} \langle c, s \rangle \right\rceil$$

- Key operation is inner product

- Want transformation that preserves inner product while reducing “size” (norm) of vectors

Bit Decomposition

- Let $\ell = \lfloor \log_2 q \rfloor + 1$ and suppose $z \in \mathbb{Z}_q^n$
- $\text{BitDecomp}(z) = (z_{1,0}, \ldots, z_{1,\ell-1}, \ldots, z_{n,0}, \ldots, z_{n,\ell-1})$ where $z_{i,j}$ is the j^{th} bit of the binary decomposition of z_i
- $\text{BitDecomp}^{-1}(z') = \left(\sum_{j=1}^{\ell} 2^j z'_{1,j}, \ldots, \sum_{j=1}^{\ell} 2^j z'_{n,j}\right)$
- $\text{PowersOfTwo}(z) = (z_1, 2z_1, \ldots, 2^{\ell-1} z_1, \ldots, z_n, 2z_n, \ldots, 2^{\ell-1} z_n)$

Bit Decomposition

- $\text{BitDecomp}(z) = (z_{1,0}, \dots, z_{1,\ell-1}, \dots, z_{n,0}, \dots, z_{n,\ell-1})$
- $\text{PowersOfTwo}(z) = \left(z_1, 2z_1, \dots, 2^{\ell-1} z_1, \dots, z_n, 2z_n, \dots, 2^{\ell-1}\, z_n\right)$

$$\langle \text{BitDecomp}(x), \text{PowersOfTwo}(y) \rangle = \langle x, y \rangle$$

Flattening a Vector

- $\text{Flatten}(z) = \text{BitDecomp}\left(\text{BitDecomp}^{-1}(z)\right)$
- $\text{Flatten}(z)$ is a 0/1 vector even though z need not be a 0/1 vector

Preserves inner product with $\text{PowersOfTwo}(\cdot)$

$$\langle x, \text{PowersOfTwo}(y)\rangle = \sum_{i=1}^{n} \sum_{j=0}^{\ell-1} x_{i,j} \cdot 2^j y_i$$

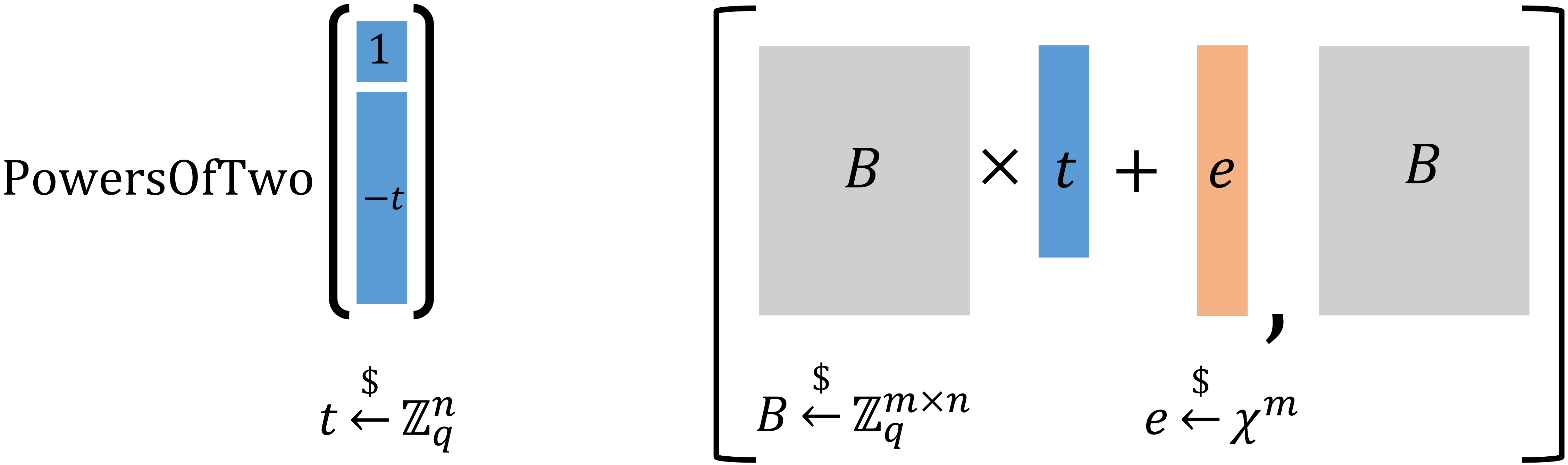
$$= \sum_{i=1}^{n} y_i \sum_{j=0}^{\ell-1} 2^j x_{i,j}$$

$$= \langle \text{BitDecomp}^{-1}(x), y\rangle$$

$$= \langle \text{Flatten}(x), \text{PowersOfTwo}(y)\rangle$$

GSW Key Generation

Regev-like, but where we apply PowersOfTwo to the secret



secret key
PowersOfTwo(s)

public key A

Note: $As = Bt + e - Bt = e$

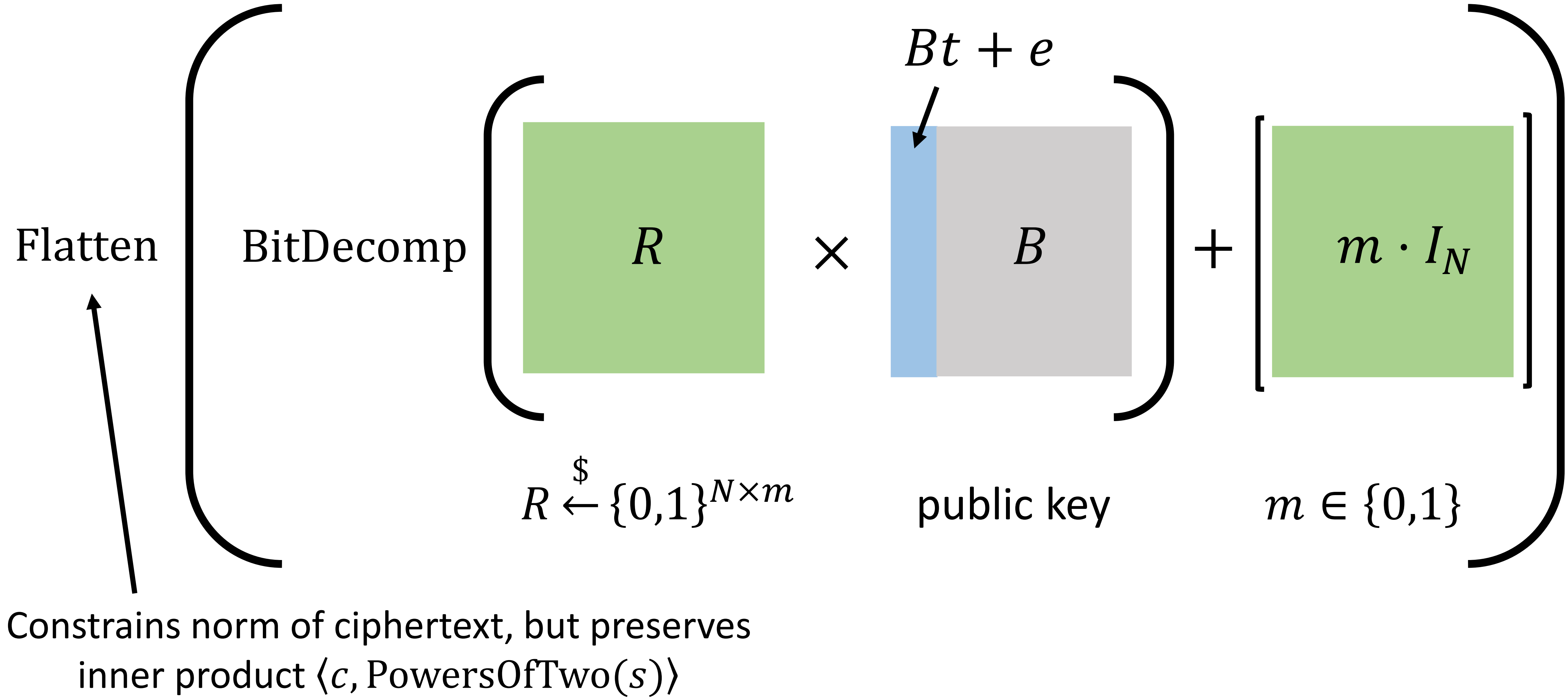
GSW Encryption

- Recall Regev decryption:

$$m \leftarrow \left\lfloor \frac{2}{q} \langle c, s \rangle \right\rceil$$

- So far, replaced s with $\mathrm{PowersOfTwo}(s)$, so to preserve inner product, we apply $\mathrm{BitDecomp}$ to the ciphertext c

GSW Encryption



Approximate Eigenvalues

- Secret key is

$$v \leftarrow \mathrm{PowersOfTwo}(s)$$

- Encryption of a message $m \in \{0,1\}$ given by

$$C \leftarrow \mathrm{Flatten}\big(m \cdot I_N + \mathrm{BitDecomp}(R \cdot A)\big)$$

- Observe:

$$Cv = mv + RAs = mv + \boxed{Re}$$

Small since R is 0/1 matrix

Revisiting Homomorphic Operations

- Homomorphic operations very natural – suppose C_1 encrypts m_1 and C_2 encrypts m_2

- Homomorphic addition: $C_1 + C_2$ encrypts $m_1 + m_2$:

$$(C_1 + C_2)v = (m_1 + m_2)v + \boxed{e_1 + e_2}$$

- If e_1 and e_2 are small, then is $e_1 + e_2$ is small

Revisiting Homomorphic Operations

- Homomorphic operations very natural – suppose C_1 encrypts m_1 and C_2 encrypts m_2

- Homomorphic multiplication: $C_1 C_2$ (almost) encrypts $m_1 m_2$:

$$C_1 C_2 v = (m_1 m_2) v + \boxed{m_2 e_1 + C_1 e_2}$$

- Noise increases based on
  - $|m_2|$: OK since $m_2 \in \{0,1\}$
  - $\|C_1\|$: OK since C_1 is 0/1 matrix

Revisiting Homomorphic Operations

- But homomorphic operations might produce matrix that is not 0/1
- Can use the Flatten operation again!

- Homomorphic addition: $\mathrm{Flatten}(C_1 + C_2)$
- Homomorphic multiplication: $\mathrm{Flatten}(C_1 C_2)$

- Ciphertext always consist of $0/1$ matrices

Brief Note on Security [High-Level]

- Public key components are simply LWE samples

- Ciphertext components are very similar to Regev encryptions (omitting a few small details, but a very similar proof carries through), and hardness derives from LWE

Questions?