# Intro to Elliptic Curves

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"Let $\mathbb{G}$ be a group of prime order."

for which discrete log and DDH is hard.

What groups do cryptographers actually use?

## Group Review

A group $(\mathbb{G}, \cdot)$ is a set $\mathbb{G}$ with a distinguished element $1$ (called the identity) and a closed, associative binary operation $\cdot : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$ such that

$\forall a \in \mathbb{G}, \exists b \in \mathbb{G}, \ a \cdot b = b \cdot a = 1$ (inverses)

$\forall a \in \mathbb{G}, \ a \cdot 1 = 1 \cdot a = a$ (identity)

Additionally, a group can be:

- abelian: if $\cdot$ is commutative, $\forall a, b \in \mathbb{G}, \ a \cdot b = b \cdot a$.
- cyclic: if $\exists g \in \mathbb{G}$ ("generator") s.t. $\mathbb{G} = \{g^0, g^1, \dots, g^{|\mathbb{G}|-1}\}$

MATH Fact: Prime order groups are abelian and cyclic!

Subgroup: a subset $\mathbb{H} \subseteq \mathbb{G}$ s.t. $(\mathbb{H}, \cdot)$ is also a group, where $\cdot$ is the natural restriction.

Order: the size of $|\mathbb{G}|$ or the least exponent $\in \mathbb{Z}$ s.t. $\underset{\in \mathbb{G}}{h}^{\mathrm{ord}(h)} = 1$.

Classic examples of prime order groups are:

- $(\mathbb{Z}_p, +)$: additive group of integers mod $p$
- a prime order subgroup of $(\mathbb{Z}_p^*, \cdot)$
  - often $p$ is a Sophie-Germain prime (also called "safe") of the form $p = 2q+1$, where $q$ is prime

<u>Discrete Log Problem (DLog)</u>

With respect to a group sampler, Sample, $\forall$ PPT $A$,

$$\Pr\left[ A(\mathbb{G}, g, g^x) = x : \begin{array}{c} (\mathbb{G}, g, p) \leftarrow \text{Sample}(1^\lambda) \\ x \xleftarrow{\$} \mathbb{Z}_p \end{array} \right] \leq \text{negl}(\lambda)$$

- DLog is ~~trivially easy~~ for $(\mathbb{Z}_p, +)$
- For $(\mathbb{Z}_p^*, \cdot)$, the best known algorithm is the General (faster than index calc) Number Field Sieve which runs in $2^{\tilde{O}((\log p)^{1/3})}$ (subexponential time).
  - For $\lambda = 128$ bits of security, $|p| \approx 3072$ bits
  - In 2019, record was a $|p| \approx 795$ bits
  - Group operations are expensive: requires arithmetic mod a 3072 bit prime

<u>Desire</u>: we would like a group that
- has an efficient group operation
- DLog, CDH, or DDH is hard
- has additional structure pairings (next class) useful for cryptography

# History of Elliptic Curves

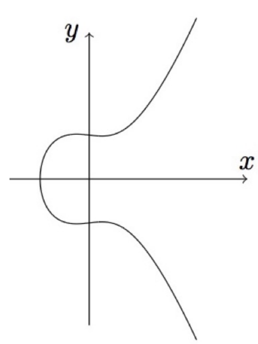
ECs are objects with deep connections to number theory and geometry. Diophantus, a greek mathematician in $3^{rd}$ century AD was interested in the set of rational points $\in \mathbb{Q} \times \mathbb{Q}$ such that $f(x, y) = 0$, for bivariate poly $f$. In general, finding rational points on plane curves can be incredibly hard.

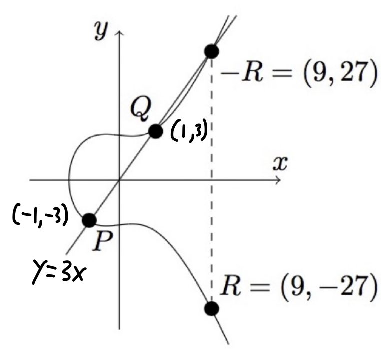
☆ Fermat wrote "Fermat's Last Thm" in the margins of Arithmetica (the series written by Diophantus on this subject)

☆ Andrew Wiles / Richard Taylor would later prove FLT using ECs – a popular book written about this by Simon Singh

In book 4 of Arithmetica, there was an excercise to find rational points satisfying $y^2 = x^3 - x + 9$.

Easy points $(0, \pm 3), (1, \pm 3), (-1, \pm 3)$



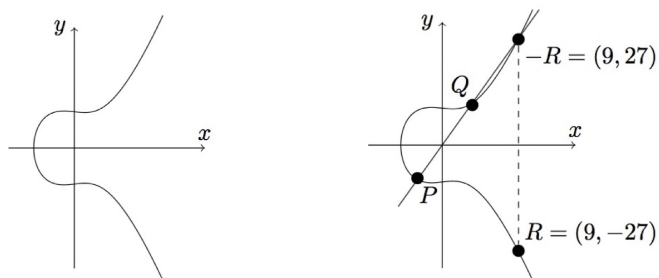

Given these easy points, can we derive other rational points?

# Elliptic Curves

an elliptic curve is a smooth plane curve defined by an equation of the form

$$y^2 = x^3 + Ax + B \quad \text{(short Weierstrass form)}$$

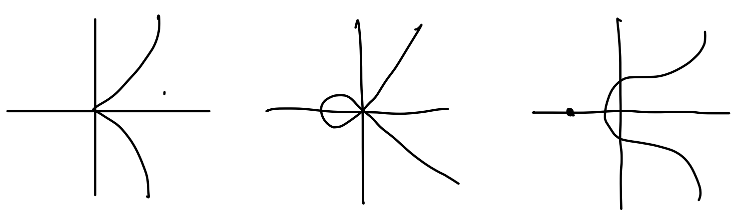
for $A, B \in \mathbb{Q}$.




$(y^2 = x^3 - x + 9)$

A curve is smooth if it has no cusps, self intersections, or isolated points.

Cannot be:



Thus, we restrict A and B such that the discriminant of the curve, $\Delta = 4a^3 + 27b^2 \neq 0$.

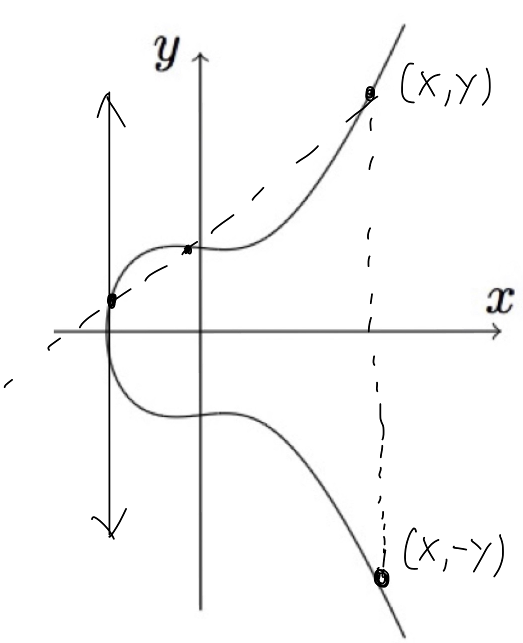
☆) This occurs when $x^3 + ax + b$ has repeated roots. $\Leftrightarrow$ when $x^3 + ax + b$ shares common roots with its derivative $3x^2 + a$.

$3x^2 + a = 0$ when $x^2 = -a/3$. $x^3 + ax + b = 0 \Leftrightarrow x(x^2 + a) + b = 0$. By substitution, $x\left(-\frac{a}{3} + a\right) + b = 0 \Leftrightarrow x = \frac{-3b}{2a}$ so when does $\underbrace{\left(\frac{-3b}{2a}\right)^2}_{\text{"}x\text{"}} = \underbrace{-\frac{a}{3}}_{\text{"}x^2\text{"}}$? $\Leftrightarrow 4a^3 + 27b^2 = 0$

**First goal:** Given some points on the elliptic curve, can we enumerate other points? (Diophantus)

## Key Observations

- An elliptic curve is symmetric: if a point $(x, y)$ is on the curve so is $(x, -y)$.
- Lines tangent to the curve at the x-axis are vertical (slope $y' = \frac{3x^2 + A}{2y}$ is $\infty$ when $y = 0$)
- A line intersecting two points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ on the curve must intersect the curve at a third point $P_3 = (x_3, y_3)$ (ignoring vertical lines) [chord method]




— Suppose $y = mx + d$ goes through $P_1, P_2$. The equation $f(x) = x^3 + Ax + B - (mx+d)^2$ has three roots $a, b, c$ giving us the three points.

* Over $\mathbb{C}$, the cubic $f(x) = (x-a)(x-b)(x-c)$ where $a, b \in \mathbb{Q}$.
$= x(x^2 - (a+b)x + ab) - c(x^2 - (a+b)x + ab) = \ldots - (a+b+c)x^2 \ldots$ Since $f(x)$ only has rational coeffs, $(a+b+c) \in \mathbb{Q}$. This implies $c$ must be rational otherwise we would get a complex/irrational coeff.

## Procedure to derive new $\mathbb{Q}$ points

- Draw a line between two points on the curve
- Obtain the point of intersection
- Flip the point across the $x$ axis to obtain a new non-collinear point.

Sneak Peek gives us an associative operation. $\longrightarrow$

# Elliptic Curve Group

What we just saw is a procedure to take two points P1 and P2 on an elliptic curve and derive a new point P3 on the curve. If we denote this operation $\boxplus$, $P1 \boxplus P2 = P3$, is the operation a group operation with the set of points being the group?

## Not Quite!

We didn't handle vertical lines nor adding points to themselves.

- Adding points to themselves can be handled by considering the line tangent to the curve at that point.
- Vertical lines requires a distinguished element called the point of infinity to be added to the group!

For an elliptic curve, $E: y^2 = x^3 + Ax + B$, we define a group $E(\mathbb{Q}) := \{\mathcal{O}\} \cup \{(x,y) \in \mathbb{Q} \mid y^2 = x^3 + Ax + B\}$ with $\boxplus$ defined as follows: (also define $-(x,y) = (x,-y)$)

- $\forall P \in E(\mathbb{Q})$, $P \boxplus \mathcal{O} = \mathcal{O} \boxplus P = P$.
- $\forall P = (x,y) \in E(\mathbb{Q}) \setminus \{\mathcal{O}\}$, define $-P = (x,-y)$ and $P \boxplus (-P) = \mathcal{O}$.
- $\forall P = (x_1, y_1), Q = (x_2, y_2) \in E(\mathbb{Q}) \setminus \{\mathcal{O}\}$, define

$$S_c = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{(chord method)}$$

$$S_t = \frac{3x_1^2 + A}{2y_1} \quad \text{(tangent method)}$$

If $P_1 \neq P_2$, $\quad x_3 = s_c^2 - x_1 - x_3 \quad y_3 = s_c(x_1 - x_3) - y_1$.

If $P_1 = P_2 \wedge y_1 \neq 0$, $\quad x_3 = s_t^2 - 2x_1 \quad y_3 = s_t(x_1 - x_3) - y_1$.

If $P_1 = P_2 \wedge y_1 = 0$, $\quad x_3 = x_1 \quad y_3 = y_1$.

Define $P \boxplus Q := (x_3, y_3)$.

Does this satisfy the properties of a group?

<u>Identity:</u> $\mathcal{O}$ ✓

<u>Inverses</u>: ✓ (the flip point)

<u>Associativity</u>: ✓ (a lot of manual algebra to prove)

Great! Can we do cryptography now?

Issues:

- Rationals don't have finite representations. This makes secure implementation hard since we don't handle infinite precision.
- hard to calculate exact order

How can we obtain a finite group of prime order using the theory of elliptic curves?

# EC over Finite Fields

$\mathbb{F}_p \cong \mathbb{Z}_p$, we will refer to as a base field.

Let $p > 3$ be a prime. An elliptic curve $E$ defined over a finite field $\mathbb{F}_p$ $(E/\mathbb{F}_p)$ is an equation

$$y^2 = x^3 + ax + b$$

where $a, b \in \mathbb{F}_p$ s.t. $4a^3 + 27b^2 \neq 0$.

this condition (the discriminant) avoids singularities

- $E(\mathbb{F}_p)$ is the set of points $(x, y) \in \mathbb{F}_p^2$ satisfying the equation and the special point at infinity $\mathcal{O}$.
- Schoof has alg running $O(\log(p^e))$ to get $|E(\mathbb{F}_{p^e})|$. ✓

Example: $E/\mathbb{F}_5 : y^2 = x^3 + 2x + 1$, $|E(\mathbb{F}_5)| = 7$

$$E(\mathbb{F}_5) = \{\mathcal{O}, (0, \pm 1), (1, \pm 2), (3, \pm 2)\}$$

Here we have a prime order group!

Note: when moving from rationals to finite fields, the properties of the addition laws needs to be reproven. This is done with a lot of algebra.

# DLog in EC Groups

Let $E/\mathbb{F}_p$ be an EC and $E(\mathbb{F}_p)$ be the group of points. Further, let $P$ be a point in $E(\mathbb{F}_p)$ of prime order $q$ $(|p| \approx |q|$ in bits$)$

$$qP := \underbrace{P \boxplus P \boxplus \ldots \boxplus P}_{q \text{ times}} = \mathcal{O}$$

$P$ must generate a prime order subgroup $(\{\mathcal{O}, P, 2P, \ldots, (q-1)P\}, \boxplus)$ of $E(\mathbb{F}_p)$.

The DLog problem is given $P$, $\alpha P$ (for random $\alpha \in \mathbb{Z}_p$), calculate $\alpha$.

- For most ECs, the best DLog attacks are $\Omega(\sqrt{q})$. This means for $\lambda = 128$ bits, the group needs to be size $\approx 2^{256}$. The group operation involves arithmetic modulo a 256-bit prime which is much faster than $(\mathbb{Z}_p^*, \cdot)$ with similar security levels. $p \neq |E(\mathbb{F}_p)| \approx p$
- There are exceptions in which DLog is easy:
  - when $|E(\mathbb{F}_p)| = p$, it is possible to map points to the additive group of $\mathbb{F}_p$ ("SMART" Attack)
  - when $|E(\mathbb{F}_p)|$ divides $p^B - 1$ for small $B$ (MOV attack)
- In practice, we standardize ECs (P256, Curve25519, etc) to use that avoids common pitfalls.

  twist secure

  Drama about parameter selection

  - either we choose an EC whose group is already a prime or pick a prime order subgroup.

    (Cauchy's theorem)

# Efficient Implementation of EC operations

- Reviewing the elliptic curve group operation, the calculation of the slope requires a field inversion

$$s = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} \\[2ex] \dfrac{3x_1^2 + A}{2y_1} \end{cases}$$

inversion

- A field inversion is much more expensive than a field addition or multiplication. Requires running a variant of the extended euclidean algo. $\approx$ 9 to 40 times a field mult (practically)
- Can we avoid field inversions when adding points?

## Jacobian Coordinates

Idea: We can "accumulate" our divisions by storing an additional element.

Let $(X : Y : Z)$ represent an affine point $\left(\frac{X}{Z^2}, \frac{Y}{Z^3}\right)$.

Affine $\mapsto$ Jacobian : $(X, Y) \mapsto (X : Y : 1)$ ,

Jacobian $\mapsto$ Affine : $(X : Y : Z) \mapsto \left(\frac{X}{Z^2}, \frac{Y}{Z^3}\right)$

Notice that when we convert to Jacobian coordinates we lose uniqueness. In particular, $\{(t^2 X, t^3 Y, t) \mid t \in \mathbb{F}\}$ all denote the same affine point $(X, Y)$.

Simarly, $\mathcal{O} \mapsto \{(t^2 : t^3 : 0) \mid t \in \mathbb{F}\}$ (i.e. $Z = 0$)

# Doubling Formula for Jacobian Coordinates

## Doubling a Jacobian $(X:Y:Z)$

$$s_t = \frac{3X_1^2 + A}{2Y} \qquad X_3 = s_t^2 - 2X_1 \quad , \quad Y_3 = s_t(X_1 - X_3) - Y_1$$

Substitute $\left(\frac{X}{Z^2}, \frac{Y}{Z^3}\right)$ into affine formulas,

$$\lambda = \frac{3\left(\frac{X}{Z^2}\right)^2 + A}{2\left(\frac{Y}{Z^3}\right)} = \frac{3X^2 + AZ^4}{2YZ}$$

$$\left.\begin{aligned} X_3 &= s_t^2 - 2\left(\frac{X}{Z^2}\right) = \frac{C}{4Y^2Z^2} \\ Y_3 &= s_t\left(\frac{X}{Z^2} - X_3\right) - \frac{Y}{Z^3} = \frac{D}{8Y^3Z^3} \end{aligned}\right\} \text{affine coords}$$

Notice $4Y^2Z^2 = (2YZ)^2$, $8Y^3Z^3 = (2YZ)^3$.

Thus, the Jacobian coords of doubling $(X:Y:Z)$ is $(C, D, 2YZ)$.

- calculation of $C$, $D$ require only a small number of field add / field mults

## Batch Conversion

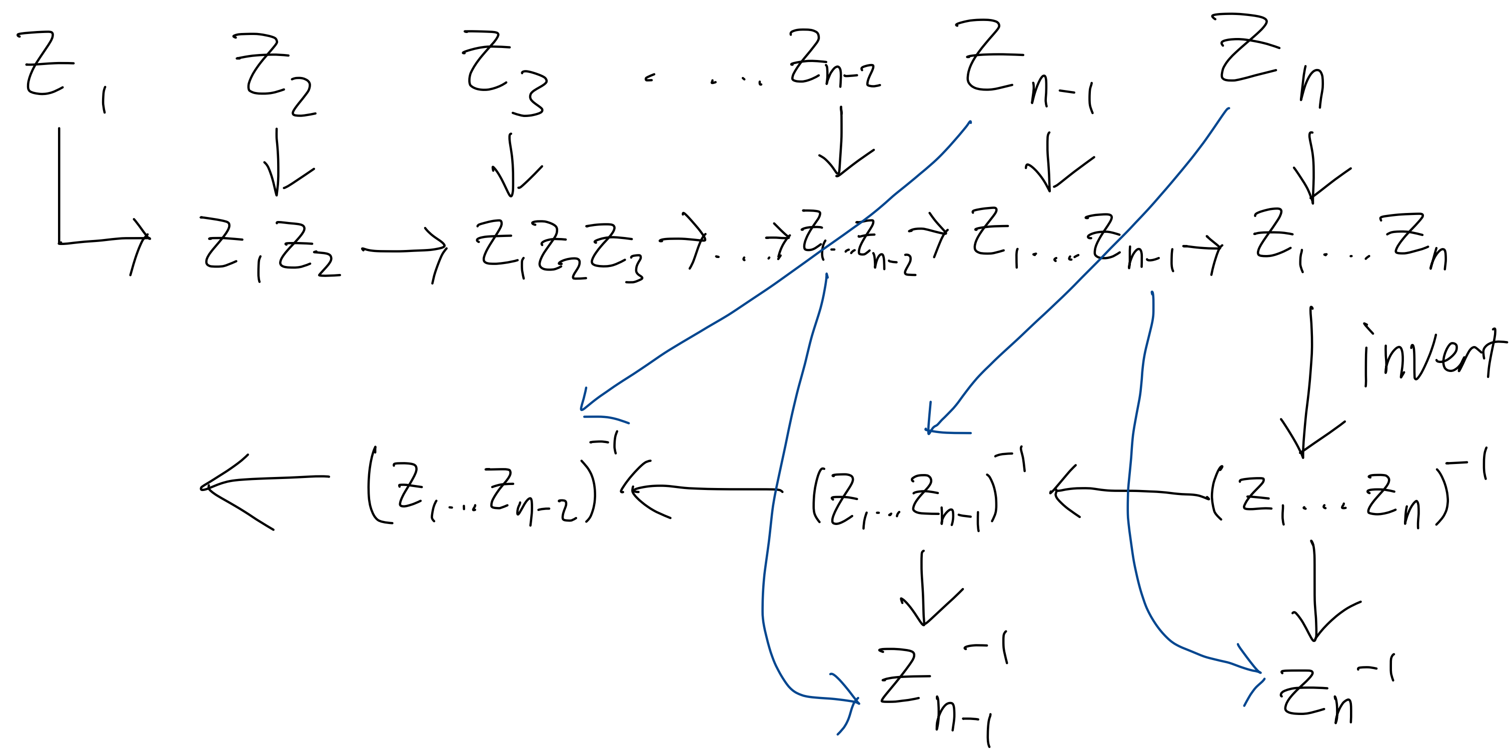
- To convert Jacobian coords to affine, we need to perform an inversion $(X:Y:Z) \mapsto \left(\frac{X}{Z^2}, \frac{X}{Z^3}\right)$, suffices to invert $Z$ and then calc $\left(\frac{1}{Z}\right)^2$, $\left(\frac{1}{Z}\right)^3$.
- Naively, to convert $n$ Jacobian points to $n$ affine points, we require $n$ inversions. However, we can batch inversions!

# Batch Inversion

- We want to invert field elts, $z_1, \dots, z_n$.
- Alg
  - Compute table of partial products
    $p := [z_1, z_1 z_2, \dots, z_1 z_2 \cdots z_n]$
  - Invert $z_1 z_2 \cdots z_n$ as $I_{1,n} := \frac{1}{z_1 z_2 \cdots z_n} = \frac{1}{p_n}$
  - $\frac{1}{z_n} = I_{1,n} \cdot p_{n-1} = \frac{1}{z_1 \cdots z_n} \cdot z_1 \cdots z_{n-1}$
  - $I_{1,n-1} = I_{1,n} \cdot z_n = \frac{1}{z_1 z_2 \cdots z_{n-1}}$
  - $\frac{1}{z_{n-1}} = I_{1,n-1} \cdot p_{n-2}$ and so on

Inverting $n$ elts requires 1 inversion, $O(n)$ mults

# Diagram



# Wrapping Up

- ECs used widely for PK crypto
- ECs are much more efficient in practice than using subgroups of $(\mathbb{Z}_p^*, \cdot)$ at similar security levels
- ECs have algebraic structure that enable many applications
  - pairings (identity based encryption, eff sigs, ..)
- most crypto libraries do not expose the group operations of ECs for safety