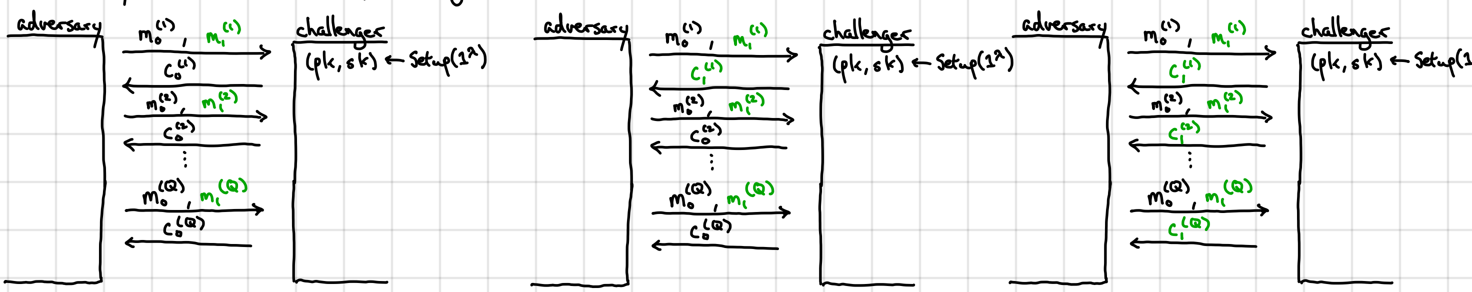
In the secret-key setting, we distinguished between semantic security and CPA-security. Here, this is <u>unnecessary</u> since

semantic security $\Rightarrow$ CPA security [means that public-key encryption must be randomized!]

↳ <u>Intuitively</u>: adversary can encrypt messages on its own (using the public key)

<u>Formally</u>: Follows from a hybrid argument



$b = 0$ [always encrypt m_0] Intermediate experiments $b = 1$ [always encrypt m_1]

Total of $Q-1$ intermediate distributions

↳ i^{th} distribution and $(i+1)^{\text{st}}$ distribution identical except on $(m_0^{(i)}, m_1^{(i)})$, challenger encrypts $m_0^{(i)}$ in distribution i and $m_1^{(i)}$ in distribution $i+1$

↳ these two distributions are indistinguishable by <u>semantic security</u> [in the reduction, the encryptions of the other messages (index $\neq i$) can be constructed using the public key (and do not depend on the challenger's choice bit)]

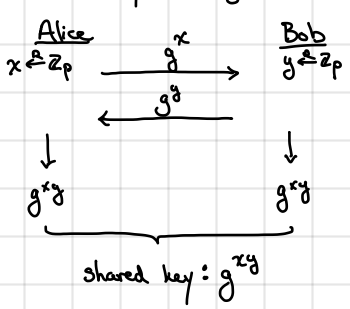
↳ if an adversary can distinguish endpoints ($b=0$, $b=1$), then it must be able to distinguish a pair of intermediate distributions [by triangle inequality]

$\therefore$ semantic security $\Rightarrow$ every pair of distributions is computationally indistinguishable

$\Rightarrow$ CPA-security

<u>PKE from DDH (ElGamal)</u>: Let $\mathbb{G}$ be a group with generator g and prime order p

Recall Diffie-Hellman key exchange:



<u>Idea</u>: Alice will publish $h = g^x$ as her public key

Bob encrypts by choosing fresh share g^y and uses g^{xy} to encrypt the message

security parameter dictates what group is used (e.g., P-256, P-384, P-512)

$$\text{Setup}(1^\lambda): \quad x \xleftarrow{R} \mathbb{Z}_p, \quad h \leftarrow g^x \qquad pk: h \quad sk: x \qquad \mathcal{M} = \mathbb{G} \quad \mathcal{C} = \mathbb{G}^2$$

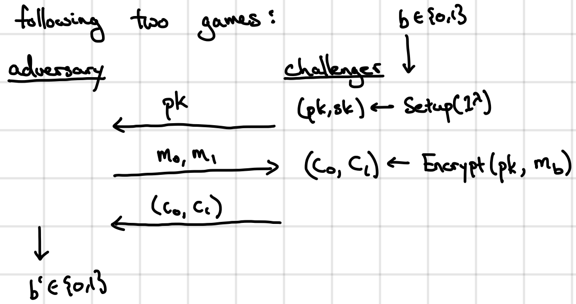
$$\text{Encrypt}(\overset{=h}{pk}, m): \quad y \xleftarrow{R} \mathbb{Z}_p, \quad c \leftarrow (g^y, m \cdot h^y)$$

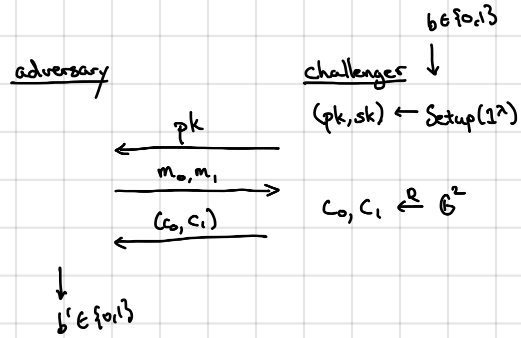
$$\text{Decrypt}(\overset{=x}{sk}, c): \quad m \leftarrow c_1 / c_0^x$$

<u>Correctness</u>:
$$\frac{c_1}{c_0^x} = \frac{m \cdot h^y}{(g^y)^x} = \frac{m \cdot (g^x)^y}{(g^y)^x} = \frac{m \cdot g^{xy}}{g^{xy}} = m$$

Security: If DDH holds in $\mathbb{G}$, then ElGamal is semantically secure.

Proof. Consider following two games:

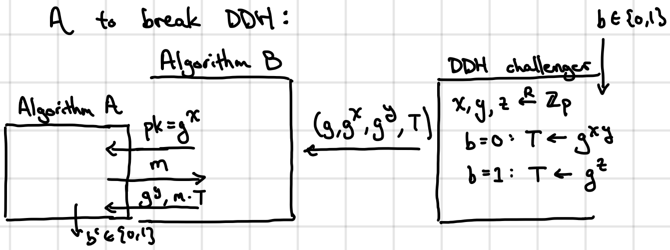




adversary's advantage in guessing b is 0 here since (c_0, c_1) is independent of (m_0, m_1)!

Claim: these two games are indistinguishable under DDH

Proof. Suppose there exists efficient $\mathcal{A}$ that can distinguish $(c_0, c_1) \leftarrow \text{Encrypt}(pk, m)$ from $(c_0, c_1) \xleftarrow{R} \mathbb{G}^2$. We use $\mathcal{A}$ to break DDH:



Observe: x is uniform over $\mathbb{Z}_p$ so g^x is a properly-generated public key (for ElGamal)

if $T = g^{xy}$, then $(g^y, T \cdot m) = (g^y, g^{xy} \cdot m)$ which is the output of Encrypt(pk, m) with randomness y — this is exactly the distribution where $\mathcal{A}$ sees Encrypt(pk, m)

if $T = g^z$, then $(g^y, g^z \cdot m)$ is uniform over $\mathbb{G}^2$ (since y, z are sampled independently of each other and of m) — this is exactly the distribution where $\mathcal{A}$ sees $(c_0, c_1) \xleftarrow{R} \mathbb{G}^2$

distinguishing advantage of $\mathcal{B}$ = distinguishing advantage of $\mathcal{A}$

Equivalent view: Under DDH, g^{xy} looks uniform even given g, g^x, g^y, so an ElGamal ciphertext looks indistinguishable (to an efficient adversary) from a OTP encryption

What if we want to encrypt longer messages? [or messages that is not a group element]

- Hybrid encryption (key encapsulation [KEM]):

  Use PKE scheme to encrypt a secret key
  Encrypt payload using secret key + authenticated encryption

  $\left.\right\}$ PKE. Encrypt(pk, k) "header" [slow] — called key encapsulation
  AE. Encrypt(k, m) "payload" [fast]

  secret-key operations much much faster than public-key operations!

- How to derive key from group element?

  Same as in key-exchange: hash the group element to a bit-string (symmetric key)

  e.g., Hash-ElGamal: Encrypt(pk, m): $y \xleftarrow{R} \mathbb{Z}_p$

  $c = (g^y, m \oplus H(g, h, g^y, h^y))$

  as before, can also rely on CDH + ideal hash function (random oracle)

  $H : \mathbb{G}^4 \to \{0,1\}^n$

Vanilla ElGamal described above is <u>not</u> CCA-secure!

Ciphertexts are malleable: given $ct = (g^y, h^y \cdot m)$, can construct ciphertext $(g^y, h^y \cdot m \cdot g)$ which decrypts to message $m \cdot g$

↳ directly implies a CCA attack

Several approaches to get CCA security from DH assumptions:

- Cramer-Shoup (CCA-security from DDH) - based on hash-proof systems
- Fujisaki-Okamoto transformation (using an ideal hash function + CDH)
- Make stronger assumption ("interactive" CDH + use ideal hash function):

We do <u>not</u> know of any groups where CDH believed to be hard, but interactive CDH is easy.

"CDH is hard even given access to a DDH oracle"

also called strong DH assumption

symmetric authenticated encryption scheme

  - Setup(1^λ): $x \xleftarrow{R} \mathbb{Z}_p$ $\quad$ pk: h
    $h \leftarrow g^x$ $\quad$ sk: x
  - Encrypt(pk, m): $y \xleftarrow{R} \mathbb{Z}_p$ $\quad k \leftarrow H(g, g^x, g^y, h^y)$ $\quad ct' \leftarrow \mathrm{Enc}_{AE}(k, m)$
    $c \leftarrow (g^y, ct')$
  - Decrypt(sk, c): $k \leftarrow H(g, g^x, c_0, c_0^x)$
    $m \leftarrow \mathrm{Dec}_{AE}(k, c_1)$

Essentially ElGamal where key derived from hash function