## Signatures from trapdoor permutations (the full domain hash):

In order to appeal to security of TDP, we need that the argument to $F^{-1}(td, \cdot)$ to be random

Idea: hash the message first and sign the hash value (often called "hash-and-sign")

↳ Another benefit: Allows signing long messages (much larger than domain size of TDF)

## FDH construction:

- Setup$(1^\lambda)$: Sample $(pp, td) \leftarrow \text{Setup}(1^\lambda)$ for the TDP and output $vk = pp$, $sk = td$
- Sign$(sk, m)$: Output $\sigma \leftarrow F^{-1}(td, H(m))$
- Verify$(vk, m, \sigma)$: Output $1$ if $F(pp, \sigma) = H(m)$ and $0$ otherwise

**Theorem.** If $F$ is a trapdoor permutation and $H$ is modeled as an ideal hash function (i.e., random oracle), then the full domain hash signature scheme defined above is secure.

↳ function that behaves like a truly random function (does not exist but functions like SHA-256 or SHA-3 believed to be "close enough")

**Proof Intuition:** To forge a signature on a message $m$, need to compute $F^{-1}(td, H(m))$ where $H(m)$ is uniformly random in domain of the permutation — security follows from TDP.

Caveat: still need to simulate signing queries ⇝ relies on "programming" the random oracle (see office hours)

## Recap: RSA-FDH signatures:

Setup$(1^\lambda)$: Sample modulus $N$, $e, d$ such that $ed = 1 \pmod{\varphi(N)}$ — typically $e = 3$ or $e = 65537$

Some (partial) attacks can exploit very small public exponent ($e=3$)

Output $vk = (N, e)$ and $sk = (N, d)$

Sign$(sk, m)$: $\sigma \leftarrow H(m)^d$ [Here, we are assuming that $H$ maps into $\mathbb{Z}_N^*$]

Verify$(vk, m, \sigma)$: output $1$ if $H(m) = \sigma^e$ and $0$ otherwise

Standard: PKCS1 v1.5 (typically used for signing certificates)

↳ Standard cryptographic hash functions hash into a 256-bit space (e.g., SHA-256), but FDH requires full domain

↳ PKCS 1 v1.5 is a way to pad hashed message before signing:

$$\boxed{\underbrace{00\ 01}_{16 \text{ bits}} \,\Big|\, \underbrace{FF\ FF \cdots FF\ FF\ 00}_{\text{pad}}\ DI \,\Big|\, H(m)}$$

DI: digest info (e.g., which hash function was used)

$H(m)$: message hash (e.g., computed using SHA-256)

↳ Padding important to protect against chosen message attacks (e.g., preprocess to find messages $m_1, m_2, m_3$ where $H(m_1) = H(m_2) \cdot H(m_3)$)
(but this is not a full-domain hash and cannot prove security under RSA — can make stronger assumption...)

Also possible to use RSA to build PKE:

"Textbook RSA" (How <u>NOT</u> to encrypt): Consider the following candidate of a PKE scheme from RSA:

- $\mathrm{Setup}(1^\lambda)$: Sample $(N, e, d)$ where $N = pq$ and $ed = 1 \pmod{\varphi(N)}$. Output $pk = (N, e)$ and $sk = (N, d)$
- $\mathrm{Encrypt}(pk, m)$: Output $c \leftarrow m^e$
- $\mathrm{Decrypt}(sk, ct)$: Output $m \leftarrow c^d$

$\Big\}$ Correct since $c^d = (m^e)^d = m^{ed} = m^1 = m \pmod N$

Correctness follows from correctness of TDP.

How about security? <u>NO</u>.

1. Security of TDP says that inverting <u>random</u> element should be difficult
   - ↳ Does not apply if messages chosen adversarially (e.g., semantic security definition)
   - ↳ Does not say anything about hiding preimage (e.g., $F(pp, x)$ can leak information about $x$ so long as leakage is not sufficient to fully recover $x$ — this is a weaker property than full indistinguishability)
2. This scheme is <u>deterministic</u>: cannot be semantically secure!

<u>NEVER</u> use textbook RSA!

To use RSA / TDPs to construct a PKE scheme, we will use a similar strategy as in the FDH signature construction:

- $\mathrm{Setup}(1^\lambda)$: Sample $(pp, td) \leftarrow \mathrm{Setup}(1^\lambda)$ for the TDP scheme and output $pk = pp$ and $sk = td$
- $\mathrm{Encrypt}(pk, m)$: Sample $x \xleftarrow{R} \mathcal{X}$ from domain of TDP — <u>Scheme is randomized!</u>

  Let $k \leftarrow H(x)$ where $H: \mathcal{X} \rightarrow \mathcal{K}$ is an (ideal) hash function and $\mathcal{K}$ is the key-space for an symmetric authenticated encryption scheme

  Compute $y \leftarrow F(pp, x)$ and $ct' \leftarrow \mathrm{Enc}_{AE}(k, m)$

  Output $(y, ct')$
- $\mathrm{Decrypt}(sk, ct' = (y, ct'))$: Compute $x \leftarrow F^{-1}(td, y)$, $k \leftarrow H(x)$, and output $m \leftarrow \mathrm{Dec}_{AE}(k, ct')$

This is an example of hybrid encryption or KEM: $y$ is used to encapsulate the key and $ct'$ is an encryption under $k$

<u>Theorem</u>. If $F$ is a trapdoor permutation and $H$ is modeled as an ideal hash function, then the above encryption scheme is semantically secure. [In fact, this scheme is CCA-secure in the random oracle model]

<u>Proof intuition</u>. Given a ciphertext $(y, ct')$ and public key $pk = pp$:

- Adversary cannot compute $x$ from $y$ (by security of TDP — since $x$ is uniform)
- Adversary cannot evaluate $H$ on $x$, so $k$ is uniformly random and hidden from adversary (if $H$ is ideal...)
- Semantic security follows from semantic security of symmetric encryption scheme.

<u>RSA instantiation</u>:

- $\mathrm{Setup}(1^\lambda)$: Sample $(N, e, d)$ where $N = pq$ and $ed = 1 \pmod{\varphi(N)}$. Output $pk = (N, e)$, $sk = (N, d)$
- $\mathrm{Encrypt}(pk, m)$: Sample $x \xleftarrow{R} \mathbb{Z}_N^*$ and compute $y \leftarrow x^e \pmod N$.

  Compute $k \leftarrow H(x)$ and compute $ct' \leftarrow \mathrm{Enc}_{AE}(k, m)$.

  $\Big\}$ Output $(y, ct')$
- $\mathrm{Decrypt}(sk, ct)$: Compute $x \leftarrow y^d \pmod N$, $k \leftarrow H(k)$, and output $m \leftarrow \mathrm{Dec}_{AE}(k, ct')$.

In practice: Most widely-used standard for RSA encryption is PKCS1 (by RSA labs)

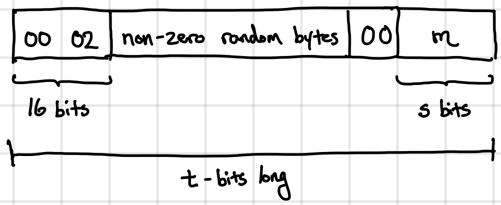
↳ Has shorter ciphertexts if we are encrypting a single $\mathbb{Z}_N$ element (no need for KEM + symmetric component]

(helpful if PKE just used to encrypt short token or metadata)

General approach: suppose N is 2048 bits and we want to encrypt 256-bit messages

↳ we will first apply a randomized padding to m to obtain a 2048-bit padded message

PKCS 1 padding:
(mode 2)




where $s \ll t$

Encryption: Compute $m_{pad} \leftarrow \text{PKCS}(m)$ and set $c \leftarrow m_{pad}^e$ [i.e., directly apply RSA trapdoor permutation to padded message]

Decryption: Compute $m_{pad} \leftarrow c^d$ and recover m from $m_{pad}$

In SSL v3.0: during the handshake, server decrypts client's message and checks if resulting $m_{pad}$ is well-formed (i.e., has valid PKCS1 padding) and rejects if not

↳ scheme is vulnerable to a chosen-ciphertext attack!

↳ allows adversary to eavesdrop on connection

Devastating attack on SSL 3.0 and very hard to fix: need to change both servers + clients!

↳ TLS 1.0: fix is to set $m \xleftarrow{R} \mathbb{Z}_N^*$ if decryption ever fails and proceed normally (never alert client if padding is malformed) — setup fails at a later point in time, but hopefully no critical information is leaked...

Take-away: PKCS1 is not CCA-secure which is very problematic for key exchange

↳ Absence of security proof should always be troubling...

New standard: Optimal Asymmetric Encryption Padding (OAEP) [1994]

↳ Can be shown to be CCA-secure in random oracle model

} Standardized in PKCS1 version 2.0