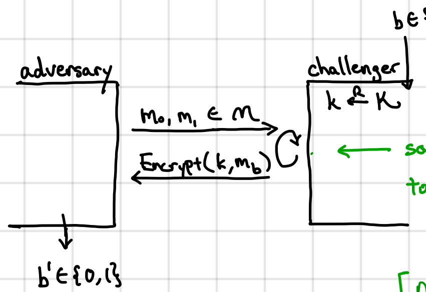
Definition: An encryption scheme $\Pi_{SE} = (\text{Encrypt}, \text{Decrypt})$ is secure against chosen-plaintext attacks (CPA-secure) if for all efficient adversaries $\mathcal{A}$:

$$\text{CPAAdv}[\mathcal{A}, \Pi_{SE}] = \left| \Pr[W_0 = 1] - \Pr[W_1 = 1] \right| = \text{negl.}$$

where W_b ($b \in \{0,1\}$) is the output of the following experiment:



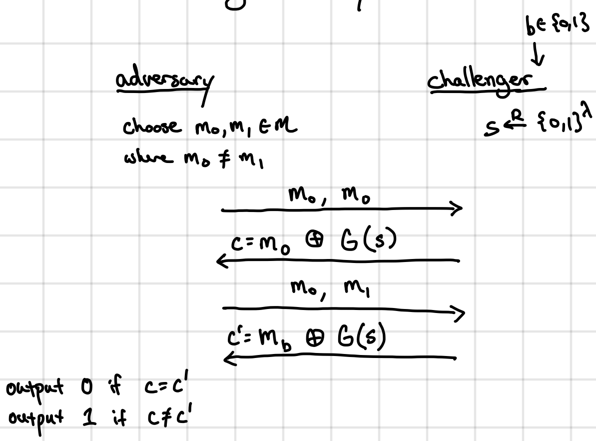
same idea as in original semantic security game, but allow adversary to make encryption queries (also called a "left-or-right" oracle)

output of experiment W_b

Adversary's goal is to guess which of m_0 or m_1 was encrypted, given access to an encryption oracle (i.e., adversary gets to see encryptions of messages of its choice.

Claim. A stream cipher is not CPA-secure.

Proof. Consider the following adversary:



$\Pr[b' = 1 \mid b = 0] = 0$ since $c' = m_0 \oplus G(s) = c$

$\Pr[b' = 1 \mid b = 1] = 1$ since $c' = m_1 \oplus G(s) \neq c$

$\Rightarrow \text{CPAAdv}[\mathcal{A}, \Pi_{SE}] = 1$

Observe: Above attack works for any deterministic encryption scheme.

$\Rightarrow$ CPA-secure encryption must be randomized!

$\Rightarrow$ To be reusable, cannot be deterministic. Encrypting the same message twice should not reveal that identical messages were encrypted.

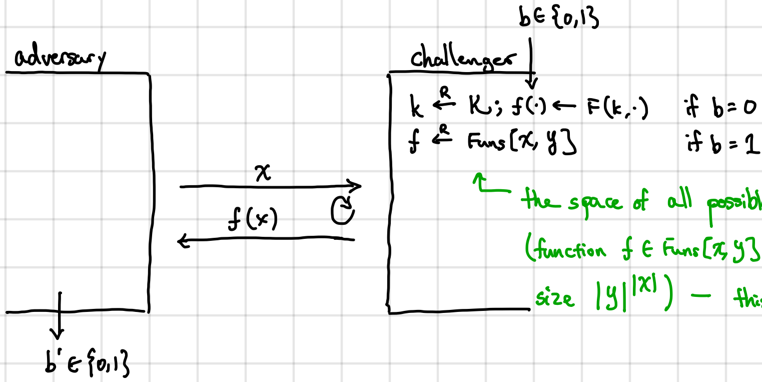
To build a CPA-secure encryption scheme, we will use a "block cipher"

- Block cipher is an invertible keyed function that takes a block of n input bits and produces a block of n output bits
- Examples include 3DES (key size 168 bits, block size 64 bits)
  AES (key size 128 bits, block size 128 bits)

Will define block ciphers abstractly first: pseudorandom functions (PRFs) and $\underbrace{\text{pseudorandom permutations (PRPs)}}_{\text{block ciphers}}$

↳ General idea: PRFs behave like random functions
PRPs behave like random permutations

Definition. A function $F: \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$ with key-space $\mathcal{K}$, domain $\mathcal{X}$, and range $\mathcal{Y}$ is a pseudorandom function (PRF) if for all efficient adversaries $\mathcal{A}$, $|W_0 - W_1| = \text{negl.}$, where W_b is the probability the adversary outputs 1 in the following experiment:



the space of all possible functions from $\mathcal{X} \to \mathcal{Y}$ (function $f \in \text{Funs}[\mathcal{X},\mathcal{Y}]$ can be represented by a truth table of size $|\mathcal{Y}|^{|\mathcal{X}|}$) — this is usually exponentially large!

$$\text{PRFAdv}[\mathcal{A}, F] = |W_0 - W_1| = |\Pr[\mathcal{A} \text{ outputs } 1 \mid b=0] - \Pr[\mathcal{A} \text{ outputs } 1 \mid b=1]|$$

Intuitively: input-output behavior of a PRF is indistinguishable from that of a random function (to any computationally-bounded adversary)

3DES: $\{0,1\}^{168} \times \{0,1\}^{64} \to \{0,1\}^{64}$ $\quad |\mathcal{K}| = 2^{168}$ $\quad |\text{Funs}[\mathcal{X},\mathcal{Y}]| = (2^{64})^{(2^{64})}$

AES: $\{0,1\}^{128} \times \{0,1\}^{128} \to \{0,1\}^{128}$ $\quad |\mathcal{K}| = 2^{128}$ $\quad |\text{Funs}[\mathcal{X},\mathcal{Y}]| = (2^{128})^{(2^{128})}$

} space of random functions is exponentially-larger than key-space!

Definition: A function $F: \mathcal{K} \times \mathcal{X} \to \mathcal{X}$ is a pseudorandom permutation (PRP) if

- for all keys k, $F(k,\cdot)$ is a permutation and moreover, there exists an efficient algorithm to compute $F^{-1}(k,\cdot)$:

$$\forall k \in \mathcal{K} : \forall x \in \mathcal{X} : F^{-1}(k, F(k,x)) = x$$

- for $k \xleftarrow{R} \mathcal{K}$, the input-output behavior of $F(k,\cdot)$ is computationally indistinguishable from $f(\cdot)$ where $f \xleftarrow{R} \text{Perm}[\mathcal{X}]$ and $\text{Perm}[\mathcal{X}]$ is the set of all permutations on $\mathcal{X}$ (analogous to PRF security)

Note: a block cipher is another term for PRP (just like stream ciphers are PRGs)

Observe that a block cipher can be used to construct a PRG:

$F: \{0,1\}^\lambda \times \{0,1\}^n \to \{0,1\}^n$ be a block cipher

Define $G: \{0,1\}^\lambda \to \{0,1\}^{\ell n}$ as

$$G(k) = F(k,1) \,\|\, F(k,2) \,\|\, \cdots \,\|\, F(k,\ell)$$

← this stream cipher allows <u>random</u> access!

↑ string concatenation

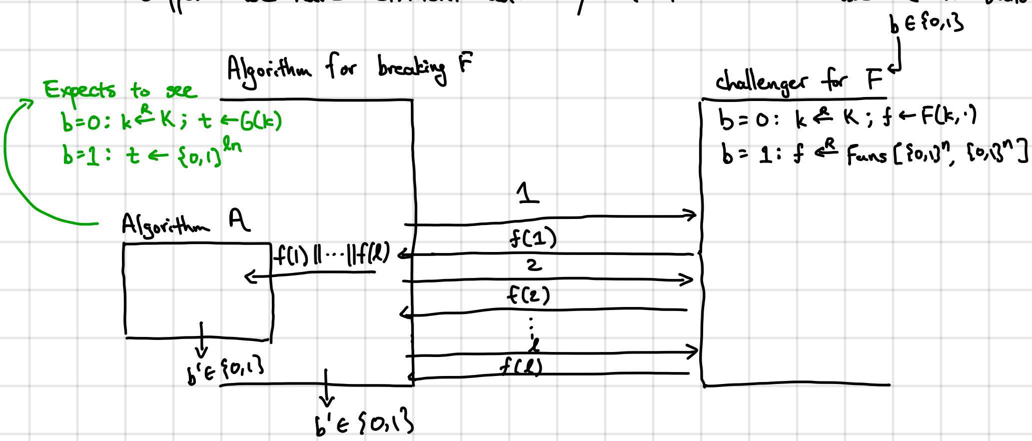
↑ write input as an n-bit string (just require that $n > \log \ell$)

we said PRP above (will revisit this)

<u>Theorem</u>. If F is a secure PRF, then G is a secure PRG.

<u>Proof</u>. As usual, we show the contrapositive: if G is not a secure PRG, then F is not a secure PRF.

Suppose we have efficient adversary A for G. We use A to build adversary for F:



1. If $\ell = \text{poly}$, then B is efficient
2. If $b=0$: B sends $G(k)$ to A where k is a uniformly random key

   If $b=1$: B sends uniformly random string (f is <u>random</u> function) to A
3. $\text{PRFAdv}[B,F] = \left|\Pr[b'=1 \mid b=0] - \Pr[b'=1 \mid b=1]\right|$

$= \left|\Pr[A \text{ outputs } 1 \mid b=0] - \Pr[A \text{ outputs } 1 \mid b=1]\right|$

$= \text{PRGAdv}[A, G]$

which is non-negligible by assumption.

But... we used a block cipher (PRP) in our construction above. Does the proof still go through?

Not quite...

for a random function, $f(1) = f(2)$ with probability $\frac{1}{2^n}$

for a random permutation, $f(1) = f(2)$ with probability 0

but 2^{-n} might be very very small... adversary won't notice unless it sees a "collision" [i.e., two values x, y where $f(x) = f(y)$]

<u>PRF Switching Lemma</u>. Let $F: K \times X \to X$ be a secure PRP. Then, for any Q-query adversary A:

$$\left|\text{PRPAdv}[A,F] - \text{PRFAdv}[A,F]\right| \leq \frac{Q^2}{2|X|}$$

<u>Proof Idea</u>. Adversary essentially cannot tell the difference unless it sees a collision. If there is no collision, then it is just seeing random values. How many queries before there is a collision? Birthday paradox: $Q \sim \sqrt{|X|}$

<u>Take-away</u>: If $|X|$ is large (e.g., exponential), then we can use a PRP as a PRF.

- 3DES: $n = 64$ so $|X| = 2^{64}$ [if adversary makes $\ll 2^{32}$ queries, then can use it as a PRF]
- AES: $n = 128$ so $|X| = 2^{128}$ [if adversary makes $\ll 2^{64}$ queries, then can use it as a PRF]

<u>Thus far</u>: PRP/PRF in "counter mode" gives us a stream cipher (one-time encryption scheme)

<u>How do we reuse it?</u> Choose a <u>random</u> starting point (called an initialization vector) "randomized counter mode"

typically, the IV is divided into a nonce (value that does not repeat) and a counter: IV = nonce ‖ counter

		m_1	m_2	m_3	m_4
$\oplus$	IV (random value)	$F(k, IV)$	$F(k, IV+1)$	$F(k, IV+2)$	$F(k, IV+3)$
	IV	c_1	c_2	c_3	c_4

divide message into blocks (based on block size of PRF)

ciphertext

<u>observe</u>: ciphertext is longer than the message (required for CPA security)

<u>Theorem</u>: Let $F: \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$ be a secure PRF and let Π_{CTR} denote the randomized counter mode encryption scheme from above for ℓ-block messages ($\mathcal{M} = \mathcal{X}^{\leq \ell}$). Then, for all efficient CPA adversaries $\mathcal{A}$, there exists an efficient PRF adversary $\mathcal{B}$ such that

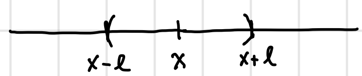
$$\text{CPAAdv}[\mathcal{A}, \Pi_{CTR}] \leq \frac{4Q^2\ell}{|\mathcal{X}|} + 2 \cdot \text{PRFAdv}[\mathcal{B}, F]$$

Q: number of encryption queries

ℓ: number of blocks in message

<u>Intuition</u>: 1. If there are no collisions (i.e., PRF never evaluated on the same block), then it is as if everything is encrypted under a fresh one-time pad.

2. Collision event: $(x, x+1, \ldots, x+\ell-1)$ overlaps with $(x', x'+1, \ldots, x'+\ell-1)$ when $x, x' \xleftarrow{R} \mathcal{X}$



probability that x' lies in this interval is $\leq \frac{2\ell}{|\mathcal{X}|}$

There are $\leq Q^2$ possible pairs (x, x'), so by a union bound,

$$\Pr[\text{collision}] \leq \frac{2\ell Q^2}{|\mathcal{X}|}$$

3. Remaining factor of 2 in advantage due to intermediate distribution (hybrid argument):

Encrypt m_0 with PRF

$\quad$ } $\text{PRFAdv}[\mathcal{B}, F] + \frac{2\ell Q^2}{|\mathcal{X}|}$

Encrypt m_0 with fresh one-time pad

$\quad$ } 0

Encrypt m_1 with fresh one-time pad

$\quad$ } $\text{PRFAdv}[\mathcal{B}, F] + \frac{2\ell Q^2}{|\mathcal{X}|}$

Encrypt m_1 with PRF

<u>Interpretation</u>: If $|\mathcal{X}| = 2^{128}$ (e.g., AES), and messages are 1 MB long (2^{16} blocks) and we want the distinguishing advantage to be below 2^{-32}, then we can use the same key to encrypt

$$Q \leq \sqrt{\frac{|\mathcal{X}| \cdot 2^{-32}}{4\ell}} = \sqrt{\frac{2^{96}}{2^{18}}} = \sqrt{2^{78}} = 2^{39}$$ (~ 1 trillion messages!)