Post-Quantum Designated-Verifier zkSNARKs from Lattices

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Argument Systems

[GMR85]

$x \in \{0,1\}^*$

accept if
$x \in \mathcal{L}$

$$\mathcal{L}_C = \{x : C(x, w) = 1 \text{ for some } w\}$$

prover

verifier

Completeness: $\forall x \in \mathcal{L}_C : \Pr[\langle P, V \rangle(x) = \text{accept}] = 1$

“Honest prover convinces honest verifier of true statements”

Soundness: $\forall x \notin \mathcal{L}_C, \ \forall \text{ efficient } P^* : \Pr\left[\langle P^*, V \rangle\left(1^\lambda, x\right) = \text{accept}\right] = \text{negl}(\lambda)$

“Efficient prover cannot convince honest verifier of false statement”

Argument Systems

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Argument system is **succinct** if:

- Prover communication is $\text{poly}(\lambda + \log|C|)$
- Running time of V is $\text{poly}(\lambda + |x| + \log|C|)$

Both must be smaller than classic NP verification

Succinct Non-Interactive Arguments (SNARGs)

[Kil92, Mic00, GW11]

$$\mathcal{L}_C = \{x : C(x, w) = 1 \text{ for some } w\}$$

prover

(x, w)

$$\pi = P(x, w)$$

Argument consists of a single message

verifier

x

accept if $V(x, \pi) = 1$

Additional properties of interest:

- **Proof of knowledge**: succinct non-interactive argument of knowledge (SNARK):

"There exists an efficient extractor that can recover a witness from any prover that convinces an honest verifier"

Succinct Non-Interactive Arguments (SNARGs)

[Kil92, Mic00, GW11]

$$\mathcal{L}_C = \{x : C(x, w) = 1 \text{ for some } w\}$$

prover

(x, w)

$$\xrightarrow{\pi = P(x, w)}$$

Argument consists of a single message

verifier

x

$\downarrow$

accept if $V(x, \pi) = 1$

Additional properties of interest:

- **Zero-knowledge**: *"Proof does not leak information about the prover's witness"*
- **zkSNARK:** zero-knowledge succinct non-interactive argument of knowledge

Succinct Non-Interactive Arguments (SNARGs)

[Kil92, Mic00, GW11]

$$\mathcal{L}_C = \{x : C(x, w) = 1 \text{ for some } w\}$$

prover

(x, w)

$\pi = P(x, w)$

verifier

x

accept if $V(x, \pi) = 1$

For general NP languages, SNARGs are
unlikely to exist in standard model [BP04, Wee05]

Succinct Non-Interactive Arguments (SNARGs)

[Kil92, Mic00, GW11]

Instantiation: “CS proofs” in the random oracle model [Mic94]

random oracle ($\mathcal{RO}$)

prover

(x, w)

$\pi = P^{\mathcal{RO}}(x, w)$

Argument consists of a single message

verifier

x

accept if $V^{\mathcal{RO}}(x, \pi) = 1$

Succinct Non-Interactive Arguments (SNARGs)

[Kil92, Mic00, GW11]

Preprocessing SNARGs: allow "expensive" setup

$\mathrm{Setup}(1^\lambda)$

Can consider publicly-verifiable and secretly-verifiable SNARGs

common reference string (CRS)

verification state

σ τ

prover

(x, w)

$\pi = P(\sigma, x, w)$

verifier

x

accept if $V(\tau, x, \pi) = 1$

Succinct Non-Interactive Arguments (SNARGs)

[Kil92, Mic00, GW11]

Very active area of research (encompassing both theory and practice):

PHGR13, BCI$^+$13, BCC$^+$16, Gro16, ZGK$^+$17, AHIV17, WTS$^+$18, GMNO18, BBB$^+$18, BBHR19, BCR$^+$19, XZZ$^+$19, LM19, CHM$^+$20, BFS20, SL20, Set20, COS20, CY21, GNS21, GMN21, GLS$^+$21, and *many, many more…*

This talk: post-quantum constructions (specifically, from lattice-based assumptions)

prover

(x, w)

$\pi = P(\sigma, x, w)$

Argument consists of a single message

verifier

x

accept if $V(\tau, x, \pi) = 1$

zkSNARK Constructions

Construction	Prover Complexity	Proof Size: Asymptotic	Proof Size: Concrete	Assumption
[Gro16]	$N \log N$	1	128 bytes	Pairings
Marlin [CHM+20]	$N \log N$	1	704 bytes	Pairings
Xiphos [SL20]	N	$\log N$	61 KB	Pairings
Pre-Quantum				
Fractal [COS20]	$N \log N$	$\log^2 N$	215 KB	Random Oracle
STARK [BBHR19]	$N \operatorname{polylog} N$	$\log^2 N$	127 KB*	Random Oracle
[GMNO18]†	$N \log N$	1	640 KB	Lattices
Post-Quantum				

†designated-verifier

*for a structured computation

Focus is on constructions with a *succinct* verifier

N: size of NP relation being verified ($N \approx 2^{20}$ for concrete values)

Asymptotic metrics are given up to $\mathrm{poly}(\lambda)$ factors (for a security parameter λ)

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				Post-Quantum

1000× gap between size of pre-quantum zkSNARKs and post-quantum ones

This talk: constructing shorter post-quantum zkSNARKs (via lattice-based assumptions)

zkSNARK Constructions

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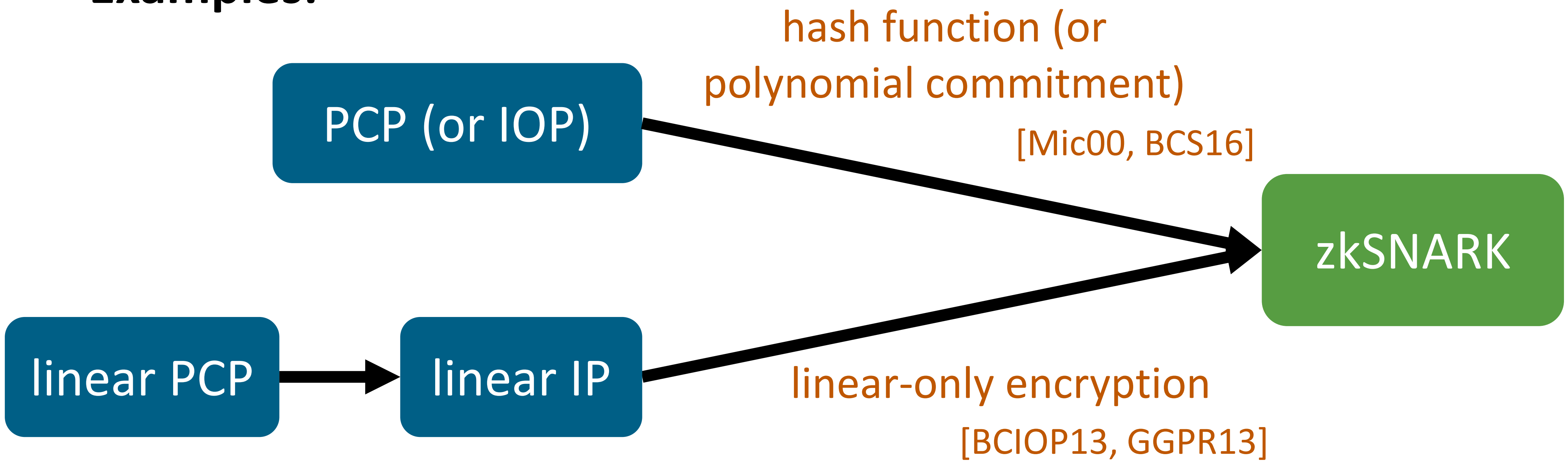
This work	$\boldsymbol{N \log N}$	**1**	**16 KB**	**Lattices**
Post-Quantum				

- ≈10× shorter proofs compared to previous post-quantum zkSNARKs for general NP relations
- Prover and verifier are concretely faster compared to most succinct pre-quantum construction [Gro16]
- Construction is designated-verifier (need secret key to check proofs) and has long CRS

Construction Overview

Follows the classic approach of combining an information-theoretic proof system (for NP) with a cryptographic compiler

Examples:

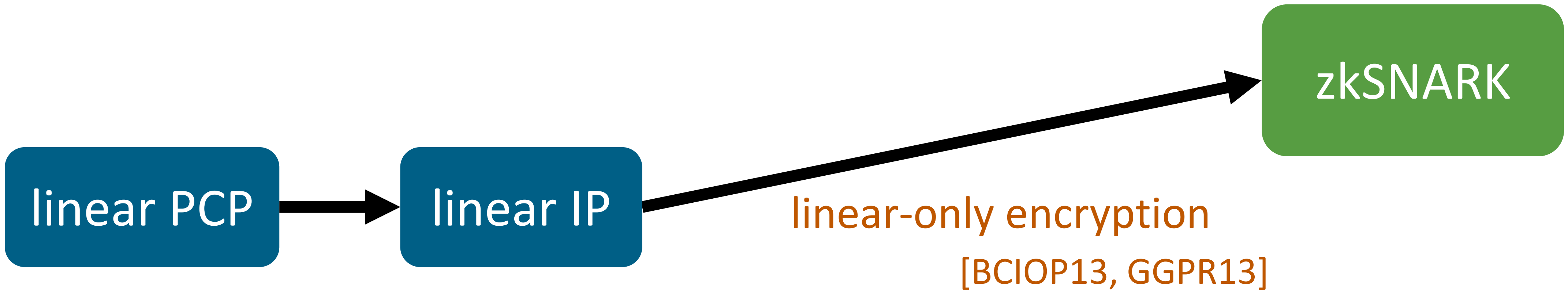


Construction Overview

Follows the classic approach of combining an information-theoretic proof system (for NP) with a cryptographic compiler

Starting point: the [BCIOP13] compiler from linear PCPs to zkSNARKs

- Yields the most succinct pre-quantum zkSNARKs [GGPR13, Gro16]
- Basis of several lattice-based zkSNARKs [BISW17, GMNO18]



Linear Probabilistically-Checkable Proofs (LPCPs)

[IKO07]

(x, w) → $\boldsymbol{\pi} \in \mathbb{F}^m$

linear PCP
"encoding" of statement/witness

$\boldsymbol{\pi} \in \mathbb{F}^m$

$\boldsymbol{q} \in \mathbb{F}^m$

$\boldsymbol{q}^{\mathrm{T}}\boldsymbol{\pi} \in \mathbb{F}$

verifier

→ accept/reject

- Verifier given oracle access to a *linear* function $\pi \in \mathbb{F}^m$
- Several instantiations:
  - 3-query LPCP based on the Walsh-Hadamard code: $m = O(|C|^2)$ [ALMSS92]
  - 4-query LPCP based on quadratic arithmetic programs: $m = O(|C|)$ [GGPR13]

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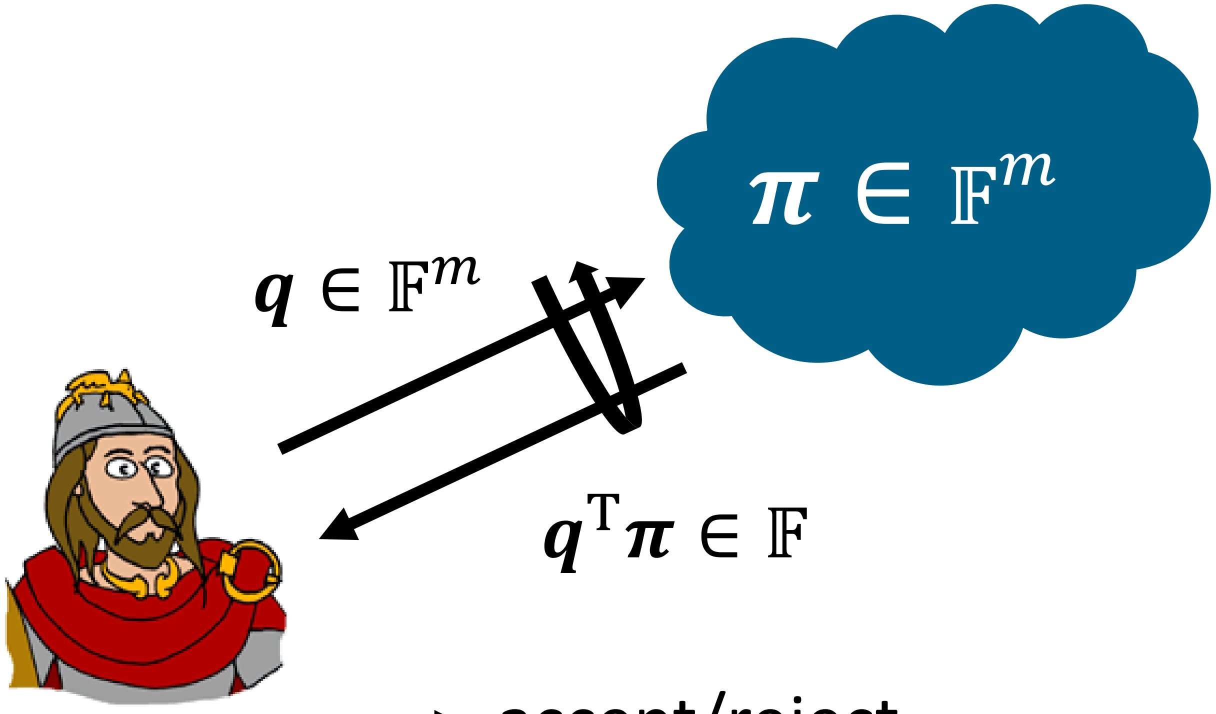

verifier → accept/reject

Oftentimes, verifier is *oblivious*:
the queries q do not depend on
the statement x

Linear Probabilistically-Checkable Proofs (LPCPs)

[IKO07]

Equivalent view (if verifier is oblivious):

$\boldsymbol{\pi} \in \mathbb{F}^m$

$\boldsymbol{Q} \in \mathbb{F}^{m \times k}$

$\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi} \in \mathbb{F}^k$

verifier

$\longrightarrow$ accept/reject

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 & \cdots & \boldsymbol{q}_k \end{bmatrix} \in \mathbb{F}^{m \times k}$$

pack all queries into single matrix

From Linear PCPs to Preprocessing SNARGs

[BCIOP13]

Oblivious verifier can "commit" to its queries ahead of time

$$\boldsymbol{Q} = \underbrace{\begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 & \cdots & \boldsymbol{q}_k \end{bmatrix}}_{\text{part of the CRS}}$$

Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

Two problems:

- Malicious prover can choose $\boldsymbol{\pi}$ based on queries
- Malicious prover can apply different $\boldsymbol{\pi}$ to the different columns of $\boldsymbol{Q}$

From Linear PCPs to Preprocessing SNARGs

[BCIOP13]

Oblivious verifier can "commit" to its queries ahead of time

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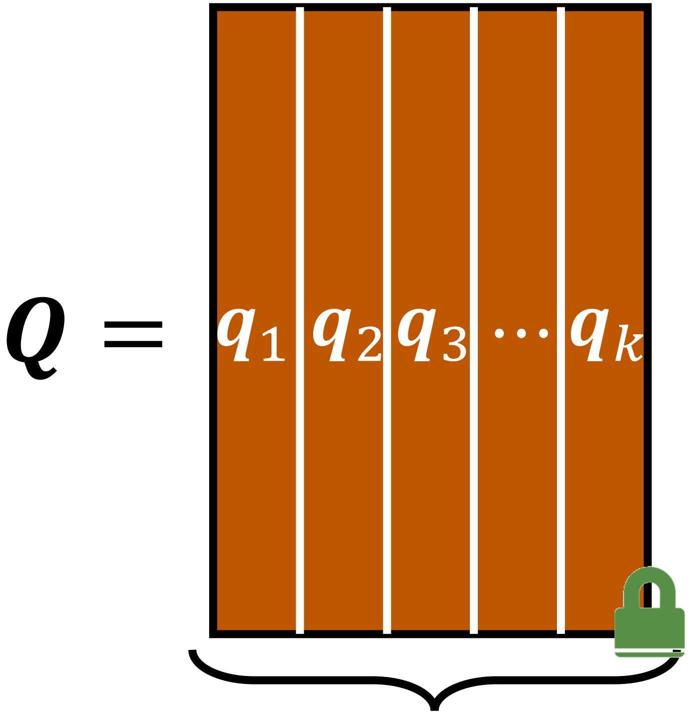
Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

Step 1: Encrypt elements of $\boldsymbol{Q}$ using additively homomorphic encryption scheme

From Linear PCPs to Preprocessing SNARGs

[BCIOP13]

Oblivious verifier can "commit" to its queries ahead of time



part of the CRS

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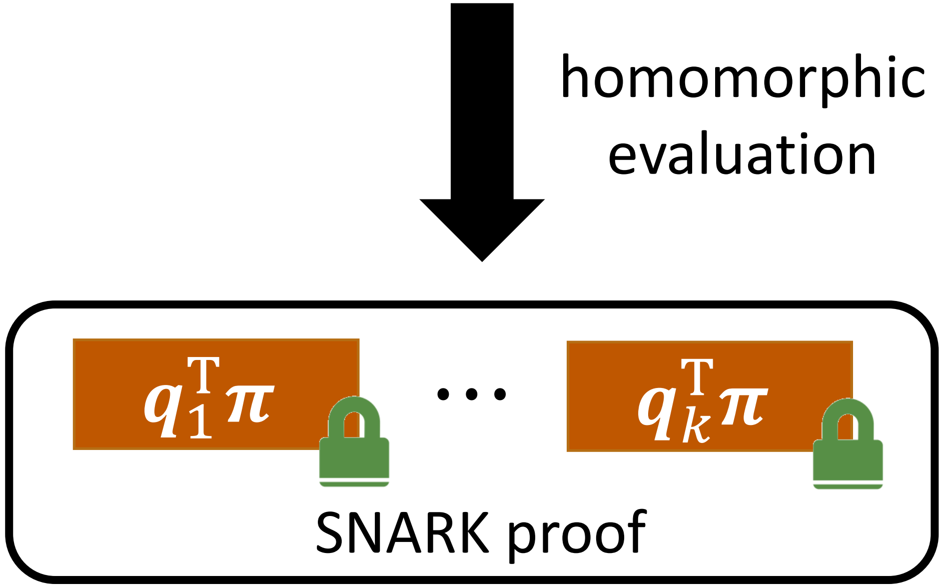

From Linear PCPs to Preprocessing SNARGs

[BCIOP13]

Designated-verifier SNARK:
decryption key needed to verify

If LPCP verification can be performed directly on ciphertexts (e.g., with pairing-based instantiations), then SNARK is **publicly-verifiable**

Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

homomorphic evaluation

$\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{\pi}$ $\cdots$ $\boldsymbol{q}_k^{\mathrm{T}}\boldsymbol{\pi}$

SNARK proof

Verifier decrypts to learn $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$ and runs linear PCP decision procedure

From Linear PCPs to Preprocessing SNARGs

[BCIOP13]

Oblivious verifier can "commit" to its queries ahead of time

$$\boldsymbol{Q} = \underbrace{\left[\, \boldsymbol{q}_1 \,\middle|\, \boldsymbol{q}_2 \,\middle|\, \boldsymbol{q}_3 \,\middle|\, \cdots \,\middle|\, \boldsymbol{q}_k \,\right]}_{\text{part of the CRS}}$$

Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

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From Linear PCPs to Preprocessing SNARGs

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Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

[BCIOP13] approach:

- Add a linear consistency check and view construction as a linear IP (LIP)
- Encrypt the LIP queries using a "linear-only" encryption scheme

From Linear PCPs to Preprocessing SNARGs

[BCIOP13]

Oblivious verifier can “commit” to its queries ahead of time

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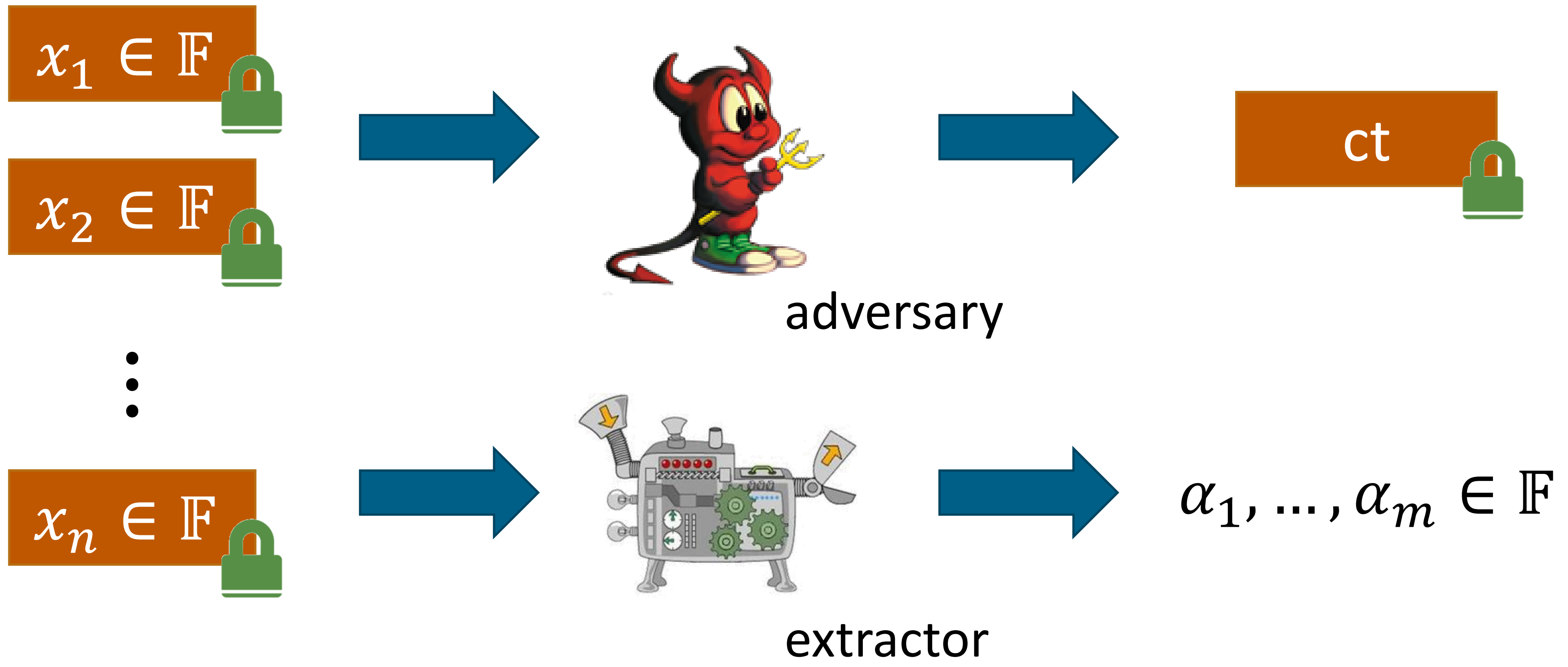
Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

Intuitively: an encryption scheme that <u>only</u> supports additive homomorphism

- Encrypt the LIP queries using a “linear-only” encryption scheme

Linear-Only Encryption

[BCIOP13]



Requirement: If $\text{Decypt}(\text{sk}, \text{ct}) \neq \perp$, then $\text{Decrypt}(\text{sk}, \text{ct}) = \sum_{i \in [n]} \alpha_i x_i$

Intuition: adversary's strategy can be "explained" by a linear function

From Linear PCPs to Preprocessing SNARGs

[BCIOP13]

Oblivious verifier can "commit" to its queries ahead of time

$$\boldsymbol{Q} = \underbrace{\begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 & \cdots & \boldsymbol{q}_k \end{bmatrix}}_{\text{part of the CRS}}$$

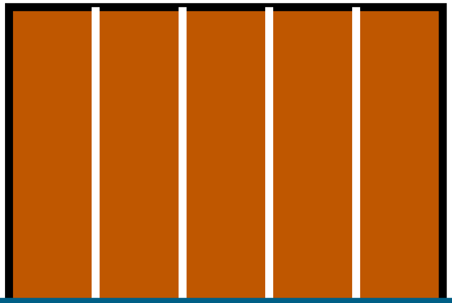
Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

All adversarial strategies can be explained by a linear function of the encrypted query components ⇒ soundness can now be based on the soundness of the linear PCP

From Linear PCPs to Preprocessing SNARGs

[BCIOP13]

Oblivious verifier can "commit" to its queries ahead of time



Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

For zero-knowledge, require that LPCP is (honest-verifier) ZK and encryption scheme is circuit private (hides linear combination)

Rest of this talk: will <u>not</u> focus on ZK

All adversarial strategies can be explained by a linear function of the encrypted query components ⇒ soundness can now be based on the soundness of the linear PCP

part of the CRS

Candidate Linear-Only Encryption from Lattices

[BISW17, GMNO18]

Conjecture: Regev encryption is linear-only

$\mathrm{KeyGen}(1^\lambda)$: Outputs a secret key $\boldsymbol{s} \in \mathbb{Z}_q^n$

$\mathrm{Encrypt}(\boldsymbol{s}, \mu \in \mathbb{Z}_p)$: Sample random $\boldsymbol{a} \leftarrow \mathbb{Z}_q^n$, error $e \leftarrow \chi$ and output

$$\mathrm{ct} = (\boldsymbol{a}, \boldsymbol{s}^{\mathrm{T}}\boldsymbol{a} + pe + \mu)$$

$\mathrm{Decrypt}(\boldsymbol{s}, \mathrm{ct})$: Write $\mathrm{ct} = (\boldsymbol{a}, b)$ and output

$$\left(b - \boldsymbol{s}^{\mathrm{T}}\boldsymbol{a} \bmod q\right) \bmod p$$

Correct as long as $|e| \leq \frac{q}{2p}$

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$$\mathrm{ct} = (\boldsymbol{a}, \boldsymbol{s}^{\mathrm{T}}\boldsymbol{a} + pe + \mu)$$

$\mathrm{Decrypt}(\boldsymbol{s}, \mathrm{ct})$:

Additive homomorphism:

- $\mathrm{ct}_1 = (\boldsymbol{a}_1, \boldsymbol{s}^{\mathrm{T}}\boldsymbol{a}_1 + pe_1 + \mu_1)$
- $\mathrm{ct}_2 = (\boldsymbol{a}_2, \boldsymbol{s}^{\mathrm{T}}\boldsymbol{a}_2 + pe_2 + \mu_2)$

Then:

$$\mathrm{ct}_1 + \mathrm{ct}_2 = (\boldsymbol{a}_1 + \boldsymbol{a}_2, \boldsymbol{s}^{\mathrm{T}}(\boldsymbol{a}_1 + \boldsymbol{a}_2) + p(e_1 + e_2) + (\mu_1 + \mu_2))$$

Homomorphic operations increase noise growth

Candidate Linear-Only Encryption from Lattices

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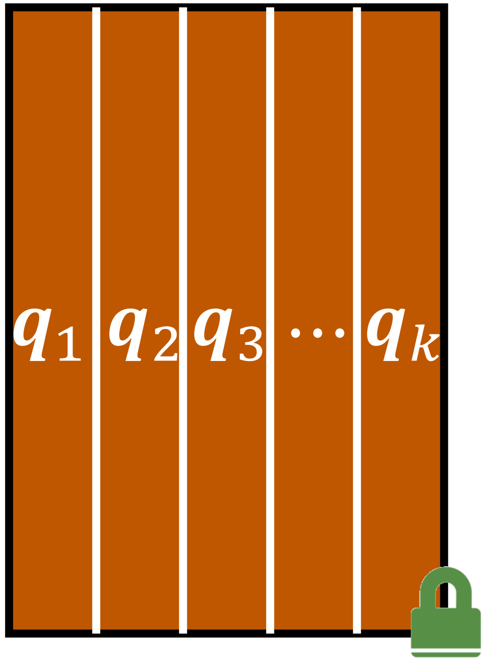
While Regev encryption can be extended to obtain FHE, existing constructions require additional components or different message embedding

Can we get more homomorphism from <u>vanilla</u> Regev?

Concrete Efficiency of Basic Instantiation

$\boldsymbol{Q} =$

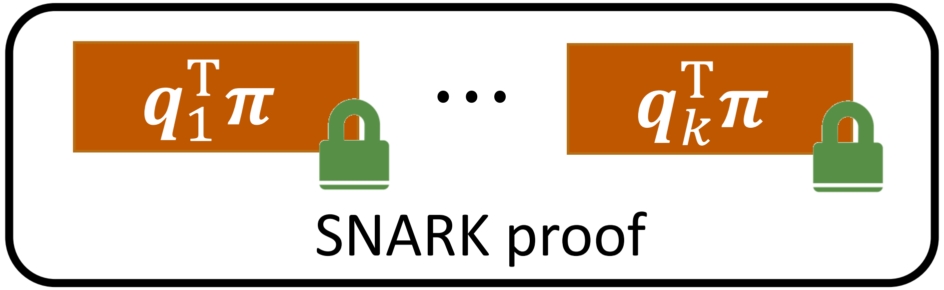


common reference string

homomorphic evaluation

linear combinations of length m over $\mathbb{F}_p$

Amount of homomorphism determines scheme parameters



Using quadratic arithmetic programs (for verifying circuit C):

- $k = 4$
- $m = O(|C|)$
- $\text{soundness} \approx \frac{2|C|}{|\mathbb{F}_p|} = \frac{2|C|}{p}$

Concrete Efficiency of Basic Instantiation

$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 & \cdots & \boldsymbol{q}_k \end{bmatrix}$

homomorphic evaluation

linear combinations of length m over $\mathbb{F}_p$

Amount of homomorphism determines scheme parameters

$\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{\pi}$ $\cdots$ $\boldsymbol{q}_k^{\mathrm{T}}\boldsymbol{\pi}$

SNARK proof

Need to choose encryption modulus q to support this amount of homomorphism:

$$q/2p > p \cdot m \cdot B$$

where B is the initial noise term

Using quadratic arithmetic programs (for verifying circuit C):

- $k = 4$
- $m = O(|C|)$
- soundness $\approx \frac{2|C|}{|\mathbb{F}_p|} = \frac{2|C|}{p}$

Concrete Efficiency of Basic Instantiation

For a circuit with $m = 2^{20}$ gates and requiring 128 bits of soundness, we require:

- $p > 2^{148}$, so $q > 2^{300}$
- At 128 bits of security, lattice dimension $n > 10^4$, so a single Regev ciphertext is *over 350 KB* (longer than other post-quantum constructions based on IOPs)
- Proof contains k ciphertexts, so proof is even longer

Alternatively: Use a small plaintext field $\mathbb{F}_p$ and amplify soundness via parallel repetition

- $p \approx 2^{20}$ and $q \approx 2^{100}$: single ciphertext is 45 KB
- Need many copies in this case (≈ 128 copies), so proof is again very long

[GMNO18]: use an instantiation where $p = 2^{32}$ *without* soundness amplification

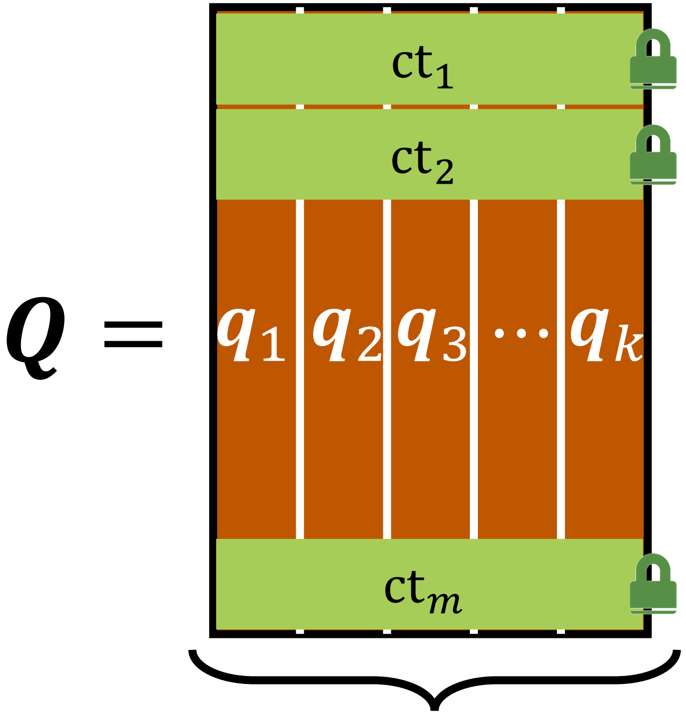
- Proofs are already 640 KB (and provide ≈ 15 bits of provable soundness for verifying computations of size 2^{16})

New techniques needed to reduce proof size

Revisiting the Bitansky et al. Compiler

[BISW17]

Oblivious verifier can "commit" to its queries ahead of time



part of the CRS

Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

Key idea: Instead of encrypting each component of $\boldsymbol{Q}$ individually, encrypt rows instead

Linear-Only Vector Encryption

[BISW17]

$v_1 \in \mathbb{F}^k$

$v_2 \in \mathbb{F}^k$

$\vdots$

$v_m \in \mathbb{F}^k$

plaintext space is a *vector* space

Linear-Only Vector Encryption

[BISW17]

$\boldsymbol{v}_1 \in \mathbb{F}^k$

$\boldsymbol{v}_2 \in \mathbb{F}^k$

$\vdots$

$\boldsymbol{v}_m \in \mathbb{F}^k$

$$\longrightarrow \quad \sum_{i \in [m]} \alpha_i \boldsymbol{v}_i \in \mathbb{F}^k$$

supports homomorphic vector addition

Linear-only: scheme only supports linear homomorphism

From Linear PCPs to Preprocessing SNARGs

[BCIOP13, BISW17]

common reference string

$$\boldsymbol{Q} = \begin{bmatrix} & & \mathrm{ct}_1 & & \\ & & \mathrm{ct}_2 & & \\ \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 & \cdots & \boldsymbol{q}_k \\ & & \mathrm{ct}_m & & \end{bmatrix}$$

Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

homomorphic evaluation

$\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

SNARK proof

Verifier decrypts to learn $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$ and runs linear PCP decision procedure

From Linear PCPs to Preprocessing SNARGs

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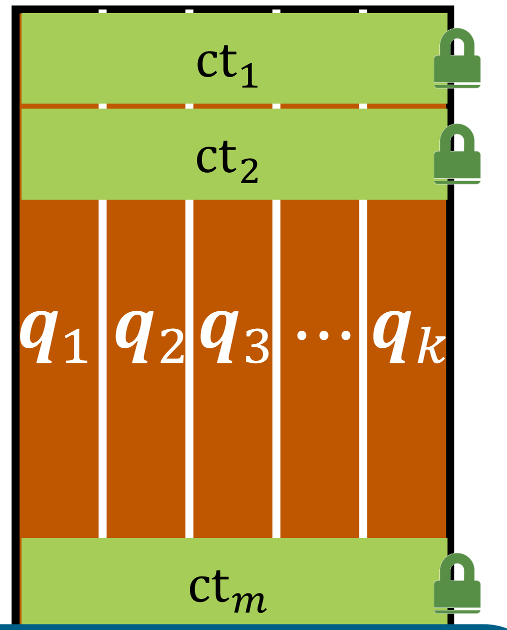
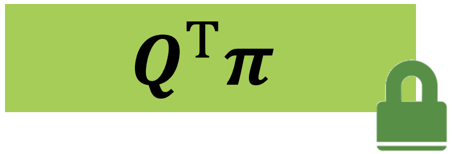


Honest prover takes (x, w) and constructs linear PCP $\boldsymbol{\pi} \in \mathbb{F}^m$ and computes $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

homomorphic evaluation



SNARK proof

- Proof is a single vector encryption ciphertext
- Allows direct compilation from linear PCPs to SNARKs (without extra linearity check from [BCIOP13])

Candidate Linear-Only Vector Encryption

[BISW17]

Conjecture: Regev encryption is linear-only

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$$\left(b - \boldsymbol{s}^{\mathrm{T}}\boldsymbol{a} \bmod q\right) \bmod p$$

Key observation: the same vector $a \in \mathbb{Z}_q^n$ can be reused with many different secret keys

Amortized/vectorized variant of Regev encryption [PVW08]

Candidate Linear-Only Vector Encryption

[BISW17]

Conjecture: Vectorized Regev encryption [PVW08] is linear-only

$\mathrm{KeyGen}(1^\lambda)$: Outputs a secret key $\boldsymbol{s} \in \mathbb{Z}_q^n$

$\mathrm{Encrypt}(\boldsymbol{s}, \mu \in \mathbb{Z}_p)$: Sample random $\boldsymbol{a} \leftarrow \mathbb{Z}_q^n$, error $e \leftarrow \chi$ and output

$$\mathrm{ct} = (\boldsymbol{a}, \boldsymbol{s}^{\mathrm{T}}\boldsymbol{a} + pe + \mu)$$

$\mathrm{Decrypt}(\boldsymbol{s}, \mathrm{ct})$: Write $\mathrm{ct} = (\boldsymbol{a}, b)$ and output

$$\left(b - \boldsymbol{s}^{\mathrm{T}}\boldsymbol{a} \bmod q\right) \bmod p$$

Candidate Linear-Only Vector Encryption

[BISW17]

Conjecture: Vectorized Regev encryption [PVW08] is linear-only

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$$|\mathrm{ct}| = (n + k) \log q$$

Would be $k(n+1)\log q$ using vanilla Regev

Ciphertext size is *additive* in the vector dimension

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Can use modulus switching [BV11, BGV12] to reduce ciphertext size after homomorphic evaluation: $(n+k)\log q \to (n+k)\log q'$

$|\mathrm{ct}| = (n +$

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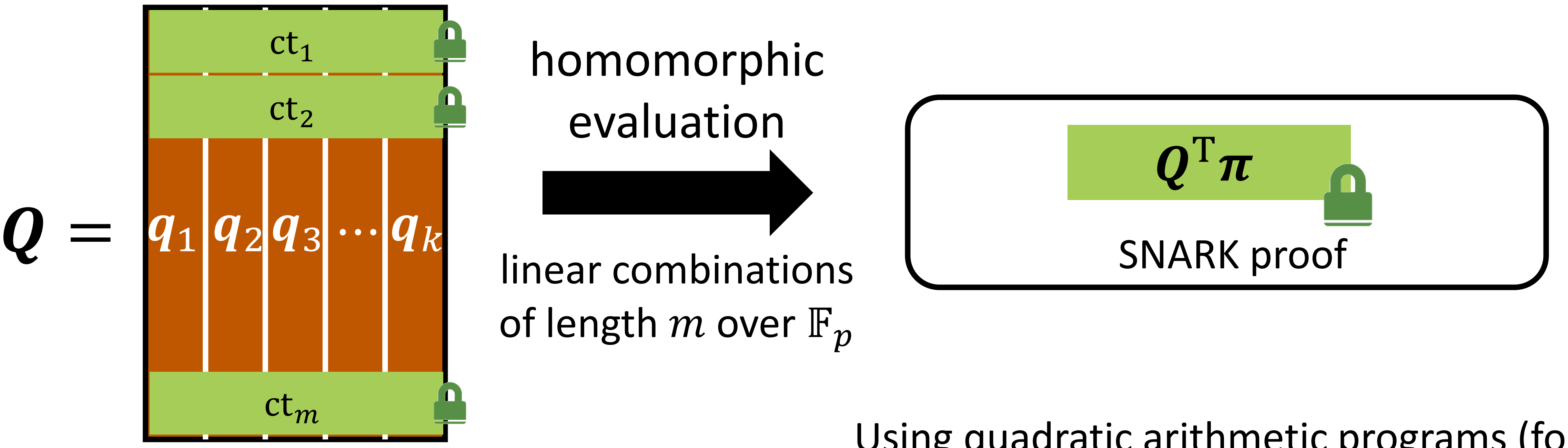
Lattice-Based zkSNARKs using Vector Encryption

[BISW17, ISW21]



common reference string

Using quadratic arithmetic programs (for verifying circuit C):

- $k = 4$
- $m = O(|C|)$
- soundness $\approx \frac{2|C|}{|\mathbb{F}_p|} = \frac{2|C|}{p}$

Lattice-Based zkSNARKs using Vector Encryption

[BISW17, ISW21]

Previously techniques to achieve small soundness:

1. Use large p (to ensure LPCP soundness); or
2. Use small p and parallel repetition to amplify soundness

Our approach: parallel repetition of LPCP to amplify soundness:

- Define LPCP to be t independent sets of queries
- Accept only if all t sets accept
- Requires kt LPCP queries and provides soundness $\left(\frac{|C|}{2p}\right)^t$

With vanilla [BCIOP13], same proof size as parallel repetition

With vector encryption, proof is always a single vector encryption ciphertext and |ct| is *additive* in vector dimension (not multiplicative)

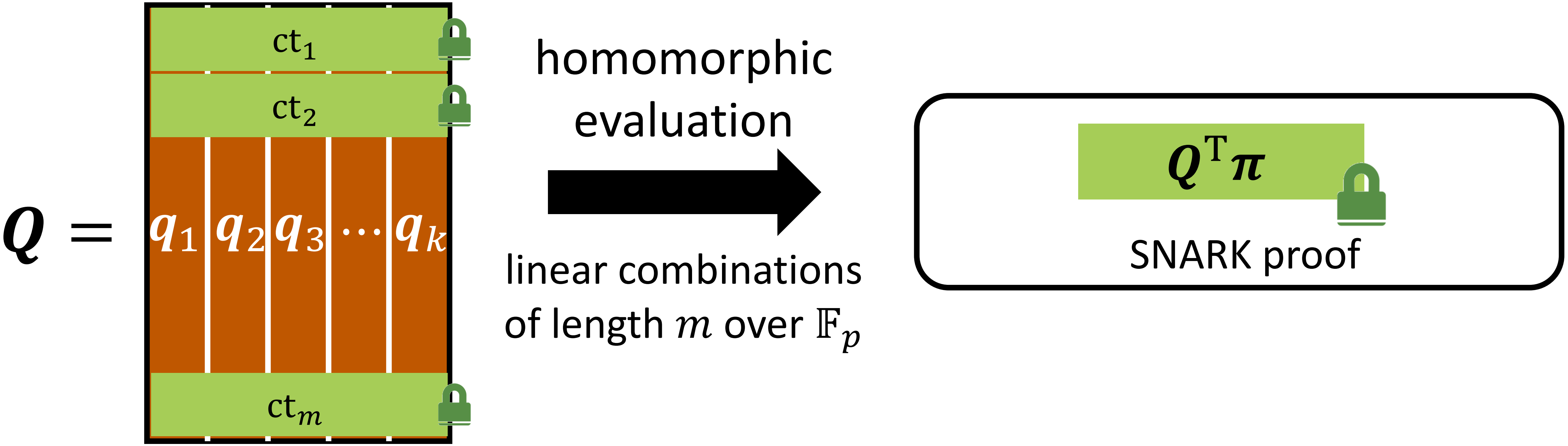
$\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\pi}$

SNARK proof

Setting $p \approx 2^{28}$, proof size is 29 KB (with a CRS of size 2.7 GB) for verifying circuit of size 2^{20}

Further Compression via Extensions Fields

[ISW21]



Can we further reduce p?

Recall: Noise growth in ciphertexts scales with

- Length m of linear combination
- Magnitude of coefficients in linear combination p

Soundness of linear PCP: $\frac{2|C|}{|\mathbb{F}|}$

Idea: use an extension field of small characteristic

Further Compression via Extensions Fields

[ISW21]

(x, w) → $\pi \in \mathbb{F}^m$

linear PCP

Suppose $\mathbb{F} = \mathbb{F}_{p^k}$ where $k > 1$

Can still instantiate using quadratic arithmetic programs

Two approaches to compile to a SNARK:

- Compile LPCP over $\mathbb{F}_{p^k}$ to a LPCP over $\mathbb{F}_p$, apply linear-only vector encryption over $\mathbb{F}_p$

  Recall that $\mathbb{F}_{p^k} \cong \mathbb{F}_p^k$; field operations in $\mathbb{F}_{p^k}$ are linear transformations over $\mathbb{F}_p^k$

  Transformation increases number of queries and query dimension by k

- Apply linear-only vector encryption over $\mathbb{F}_{p^k}$

  Work over a polynomial ring $R = \mathbb{Z}[x]/\Phi_m(x)$ where m is chosen so that $R/pR \cong \mathbb{F}_{p^k}$

  Consider Regev encryption over R (using module lattices)

Further Compression via Extensions Fields

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In both settings: coefficients of prover's linear combination have magnitude $\approx p$ while field has size p^k

Further Compression via Extensions Fields

[ISW21]

$(x, w) \longrightarrow \pi \in \mathbb{F}^m$

linear PCP

This work: consider quadratic extension fields

- $R = \mathbb{Z}[x]/(x^2 + 1)$ and set $p = 3 \bmod 4$ so $R_p = R/pR \cong \mathbb{F}_{p^2}$
- Choose ciphertext q to be a power of 2
  - All arithmetic operations can be implemented using 128-bit arithmetic
  - Low degree means polynomial arithmetic only slightly more expensive

Further Compression via Extensions Fields

[ISW21]

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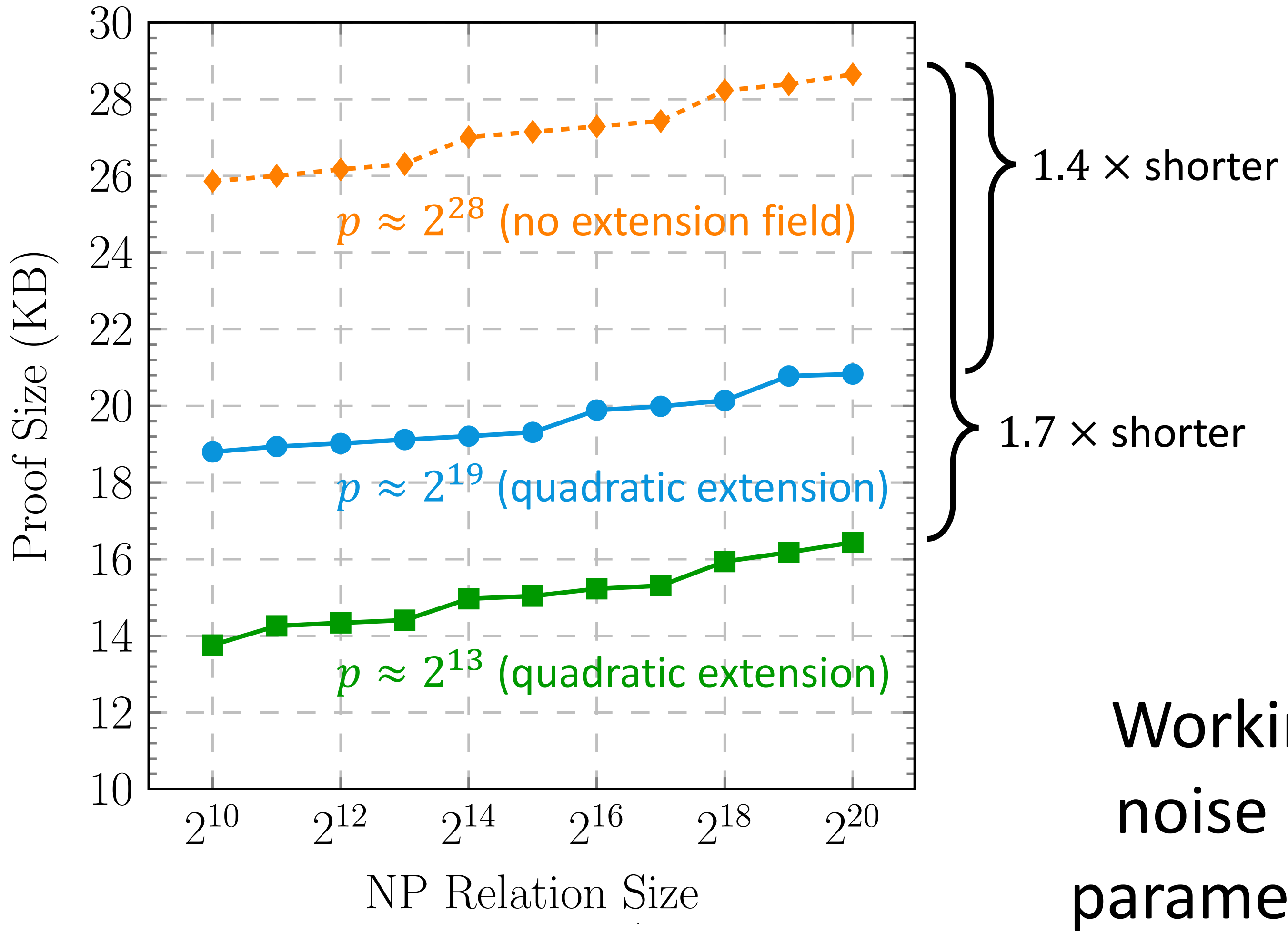
- Choose ciphertext q to be a power of 2
  - All arithmetic operations can be implemented using 128-bit arithmetic
  - Low degree means polynomial arithmetic only slightly more expensive
- Choose $p = 2^t \pm 1$ so $\mathbb{F}_{p^2}$ has 2^{t+1}-th roots of unity (for efficient implementation of LPCP prover)

Higher-degree extension makes polynomial arithmetic more costly (or need non-power-of-two modulus to exploit FFTs)

Further Compression via Extensions Fields

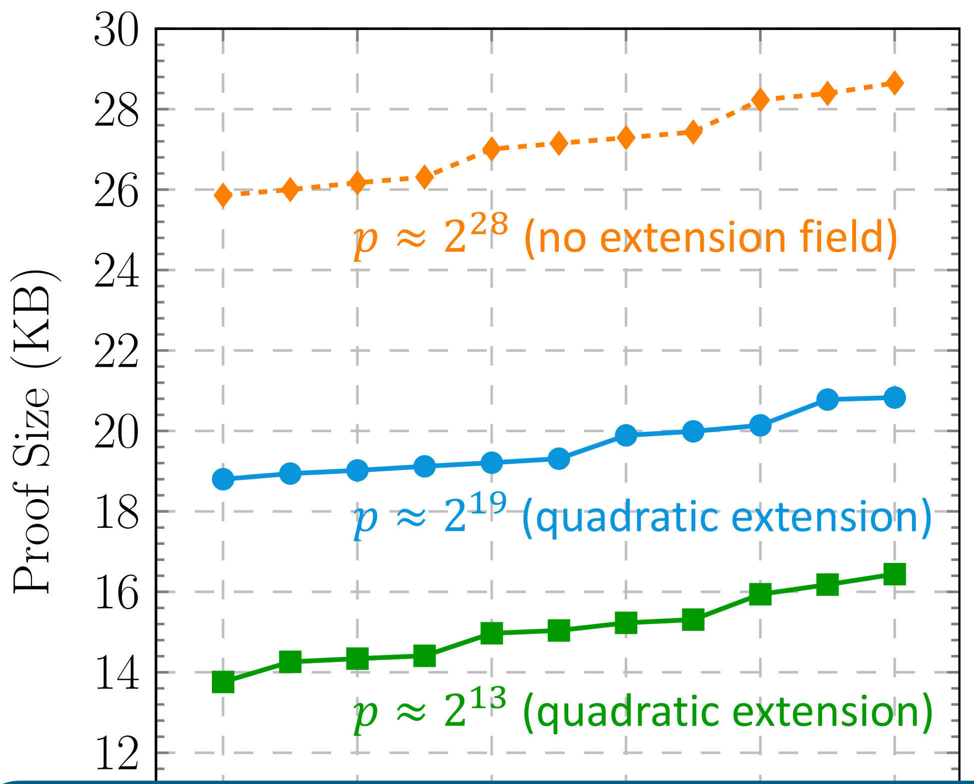
[ISW21]

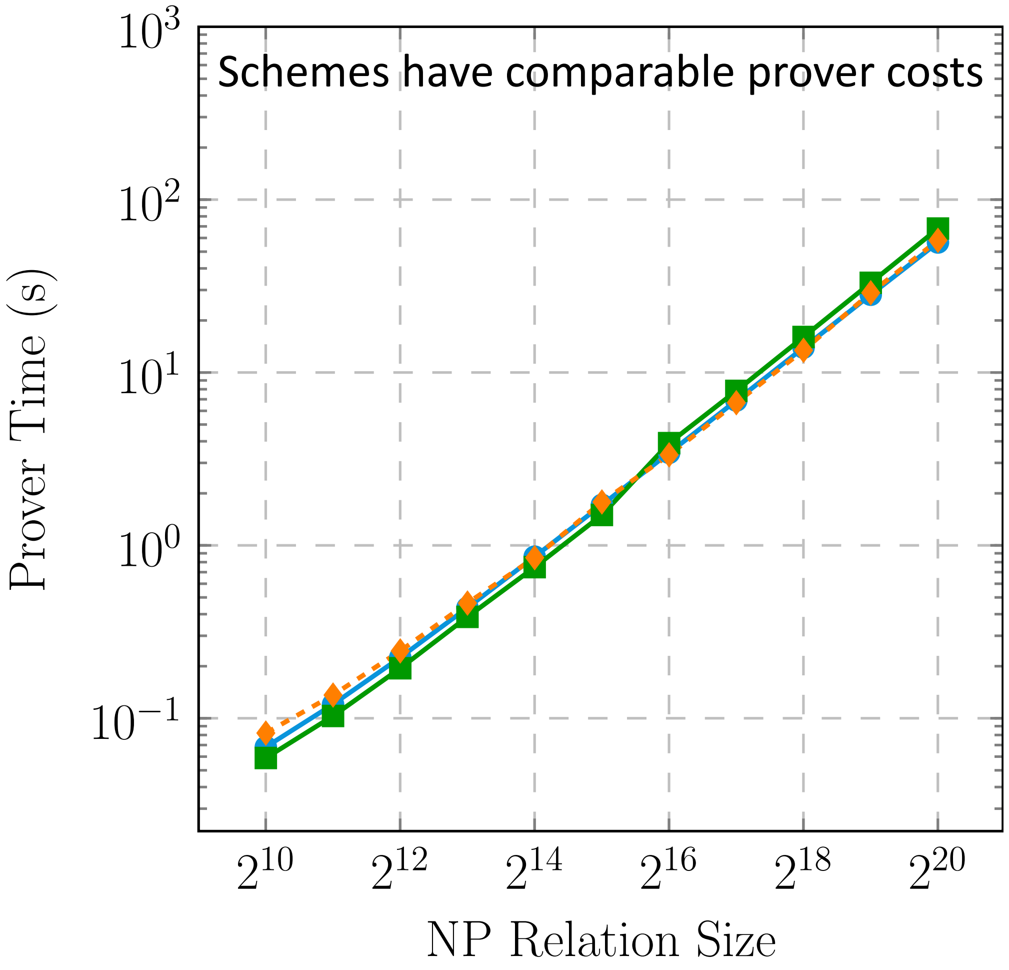


Working over extension field reduces noise accumulation ⇒ smaller lattice parameters ⇒ concretely shorter proofs

Further Compression via Extensions Fields

[ISW21]

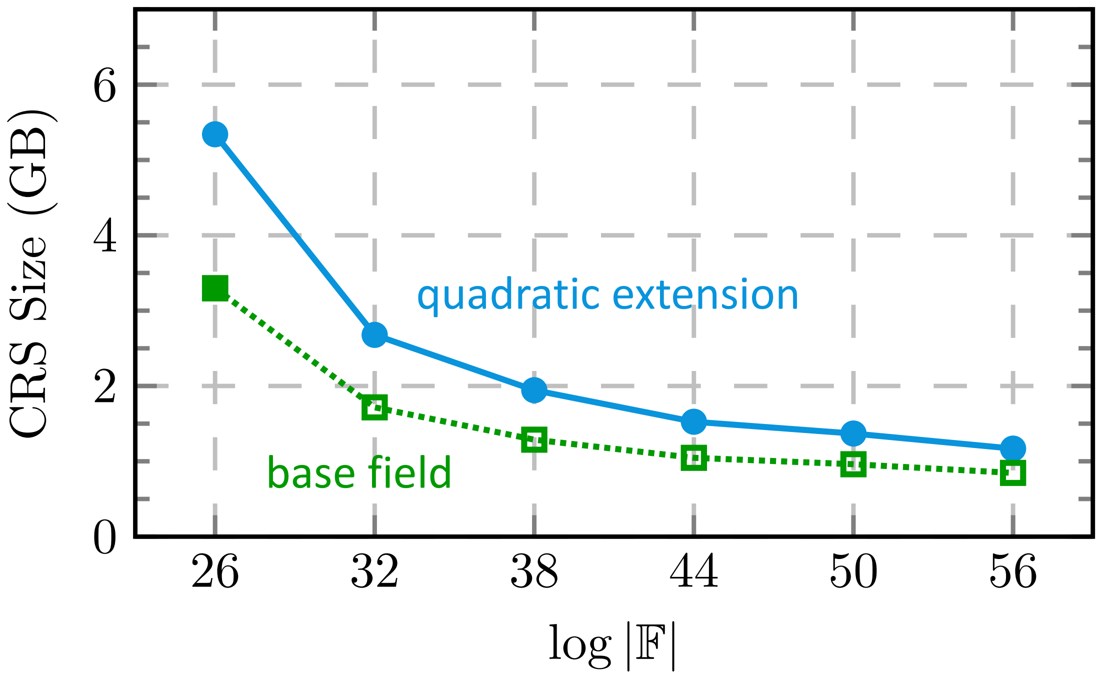


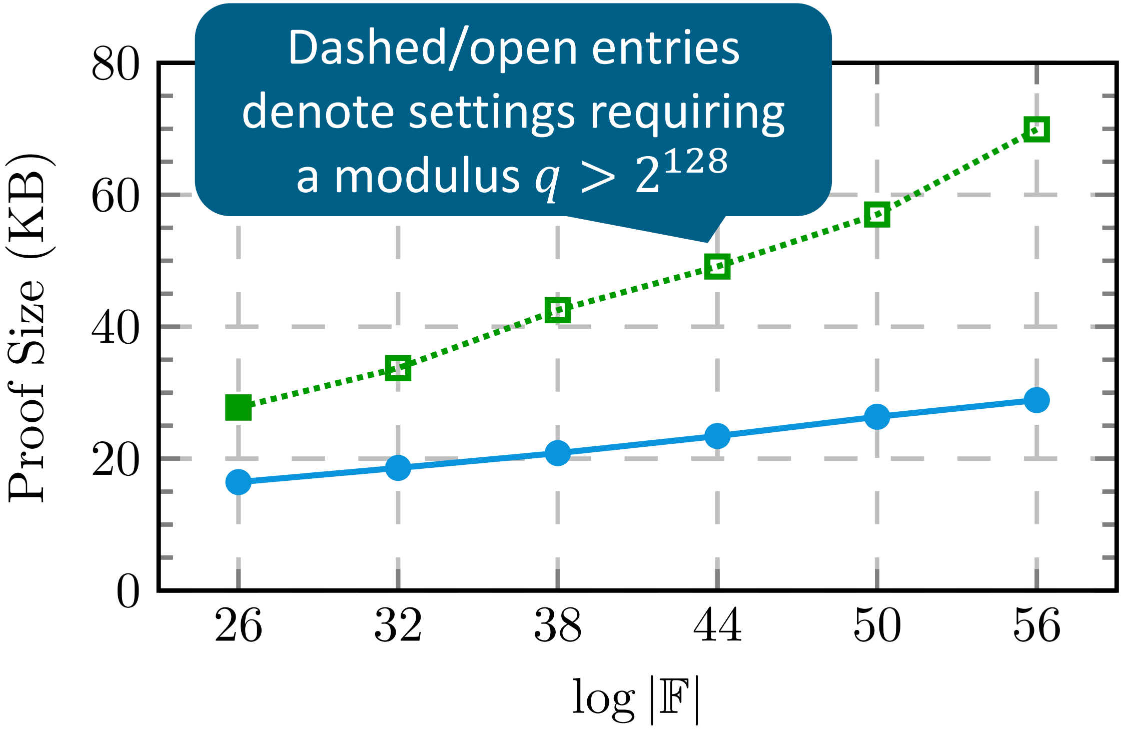


- Slightly more expensive homomorphic operations over extension field, but smaller lattice parameters
- Smaller field ⇒ more LPCP queries for soundness amplification ⇒ higher prover cost

Effect of Field Size

[ISW21]





Using the extension field increases CRS size but decreases proof size

- CRS consists of "compressed" ciphertexts where random component is derived from a PRF (i.e., $\mathrm{ct} = (\boldsymbol{a}, \boldsymbol{v})$ where $\boldsymbol{a}$ is random and $v = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{a} + p\boldsymbol{e} + \boldsymbol{\mu}$)
- Proof consists of full ciphertexts

[see paper for more microbenchmarks]

Concrete Comparison with zkSNARKs

[ISW21]

	Size		Time			
Construction	**CRS**	**Proof**	**Setup**	**Prover**	**Verifier**	**Assumption**
[Gro16]	199 MB	128 bytes	72 s	79 s	3.4 ms	Pairings
						Pre-Quantum
Ligero [AHIV17]	–	14 MB	–	38 s	22 s	Random Oracle
Aurora [BCR+19]	–	169 KB	–	304 s	6.3 s	Random Oracle
Fractal [COS20]	11 GB	215 KB	116 s	184 s	9.5 ms	Random Oracle
This work	**5.3 GB**	**16.4 KB**	**2240 s**	**68 s**	**1.2 ms**	**Lattices**
This work	**1.9 GB**	**20.8 KB**	**877 s**	**56 s**	**0.4 ms**	**Lattices**
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All benchmarks collected on same hardware for verifying NP relation of size 2^{20}

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Over 10.3× shorter than other post-quantum SNARKs

Still over 131× longer than pairing-based SNARKs

Over 42× shorter than previous lattice-based SNARKs [GMNO18] (based on reported numbers for verifying circuit of size 2^{16})

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Prover cost is essentially cost of LPCP prover and computing a linear combination

1.2× faster than pairing-based SNARKs

Slower than schemes like Ligero based on MPC-in-the-head (which does not have succinct verification)

If we consider restricted computations, can have much faster provers (e.g., ethSTARK [BBHR19])

All benchmarks collected on same hardware for verifying NP relation of size 2^{20}

Concrete Comparison with zkSNARKs

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Lattice-based SNARKs have very lightweight verification: computing a matrix-vector product and rounding

Well-suited for lightweight or energy-constrained devices

All benchmarks collected on same hardware for verifying NP relation of size 2^{20}

Concrete Comparison with zkSNARKs

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Limitations of lattice-based SNARKs:

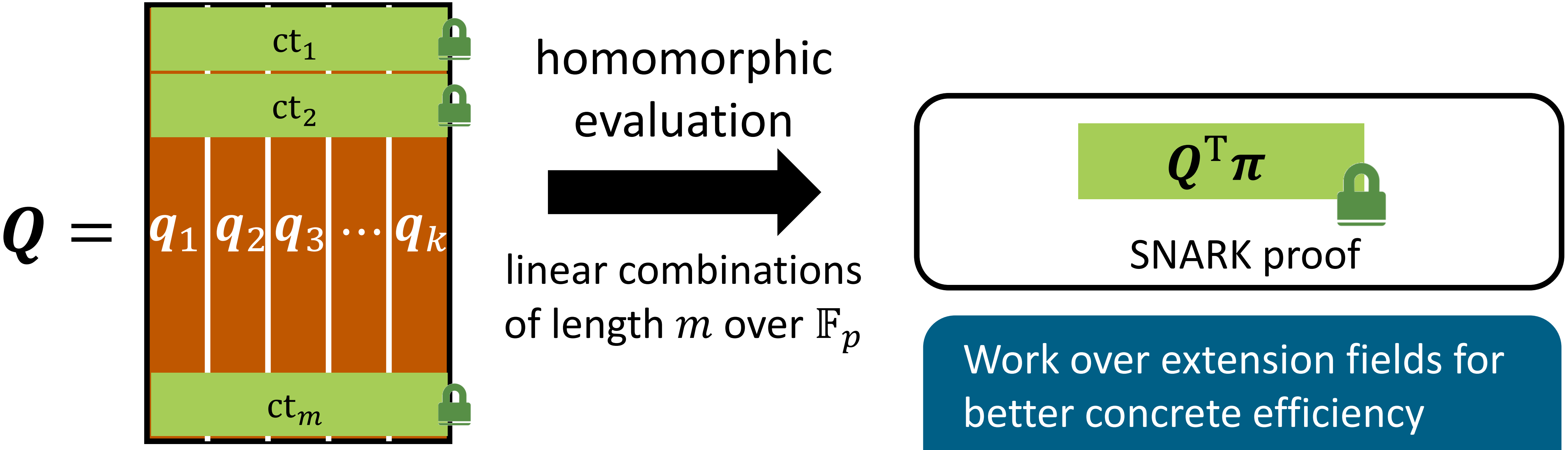
- Require expensive trusted setup (need to encrypt large number of vectors)
- Resulting construction is designated-verifier (other schemes are publicly-verifiable)
- Resulting CRS is large (lattice ciphertexts still large, even with compression)

All benchmarks collected on same hardware for verifying NP relation of size 2^{20}

Summary

Directly compile linear PCPs to SNARKs using linear-only vector encryption

Instantiate linear-only vector encryption from vectorized Regev encryption



Work over extension fields for better concrete efficiency

[see paper for further optimizations]

Open Problems

Publicly-verifiable SNARKs from lattice-based assumptions

Constructions with short proofs but expensive verifiers are known from lattices [BBC+18, BLNS20]

Concretely-efficient designated-verifier SNARKs with **reusable soundness** from lattices

Thank you!

https://eprint.iacr.org/2021/977

https://github.com/lattice-based-zkSNARKs/lattice-zksnark